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APRIL 1957

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VERA SANFORD

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Robert Recorde's Whetstone of witte, 1557

VERA SANFORD, *State Teachers College, Oneonta, New York.*

There is little that is original in the content of Robert Recorde's Whetstone of witte. *Yet this book contains the "pair of twin paralleles of one lengthe" which Recorde used to mean "is equalle to." This was his own invention*

THE YEAR 1957 is the four hundredth anniversary of the publication of Robert Recorde's *Whetstone of witte*, the first algebra printed in English, the book in which the equality sign (=) was used for the first time. It is appropriate to mark this anniversary by quoting Recorde's explanation of his invention and by outlining its later history.

Contrary to the natural assumption that the value of this symbol would have been recognized at once, it develops that fully sixty years elapsed before its second appearance, although in less than a century after that time, it had general acceptance. This situation suggests that we examine the state of mathematics in England in the 1550's, that we review what we know about the author, and study the contents of the book itself.

I. THE EQUALITY SIGN

At the beginning of his work with equations, Recorde says

> And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke vse, a pair of paralleles, or Gemowe (twin) lines of one lengthe, thus: ═══ bicause noe .2. thynges can be moare equalle.

The equality sign is Recorde's single contribution to the symbolism of algebra. Although the signs + and − were already in use, having letters represent unknown quantities was in the future. Algebra looked very different from the way it looks today.

In writing equations, Recorde's contemporaries used the word *equal* in its different forms as "aequales," "faciunt," "gleich." Sometimes they abbreviated these words. There was considerable experimentation with symbols, governed in part by the type the printers had on hand. In the years following the publication of the *Whetstone of witte*, different writers used a number of symbols to represent equality and the symbol (=) stood for a variety of meanings.[1] In view of this competition and in consideration of the confusion of meanings assigned to the twin parallels, it is all the more remarkable that Recorde's invention ultimately won universal adoption. The process was slow.

The second appearance of the equality sign was in an appendix to a translation (1618) of Napier's first work on logarithms. It is supposed that this appendix was the work of William Oughtred, whose *Clavis mathematicae* (1631), a closely-packed work on arithmetic and algebra, was influential in winning for Recorde's symbol a general acceptance in England in the seventeenth century. By the eighteenth century, it had come into use on the continent.

[1] Florian Cajori, *History of Mathematical Notations* (Chicago, 1928), Vol. I.

II. Robert Recorde and his times

Robert Recorde (*c.* 1510–1558) was born shortly after Henry VIII came to the throne of England. He died just before the accession of Elizabeth I.

His contemporaries on the continent included Cardan and Tartaglia who were doing pioneer work in mathematics in Milan and Venice. There were the German algebraists Rudolff, Stifel, and Scheubel. Copernicus, from Poland, published his great work in 1543. In 1538, Mercator, in Flanders, produced a map of the world on a projection used earlier by a Spaniard, but now called by his name. Methods of navigation were being improved. Almanacs of a sort were being published. The variation and the dip of the magnetic needle were being studied. The problem of the determination of longitude baffled mathematicians and other scientists. Sebastian Cabot maintained that he had solved it, but he died in 1557 without divulging his secret, and it was supposed that his boasts were the imaginings of an octagenarian.

England lagged behind in this activity. In fact the one important publication in mathematics prior to Recorde was Cuthbert Tunstall's *De arte Svppvtandi* (1522), a scholarly arithmetic in Latin based largely on continental models. So far as navigation and exploration were concerned, it is true that in 1497 the king had financed the voyage of the Genoese mariners John and Sebastian Cabot to the New World. John Cabot died the next year. Royal support stopped at this point, and Sebastian Cabot entered the employ of the King of Spain.

It was in the first half of the sixteenth century, however, that the center of world trade began to shift. Prior to this time, goods from China, India and the East Indies had been brought by caravan to the ports on the eastern shores of the Mediterranean where they were loaded on ships from Genoa and Venice and other Italian cities for distribution to different parts of Europe. The Mediterranean was still the great highway it had been in the days of Greece and Rome.[2] But in the period under consideration, sea routes were being developed. England no longer lay at the periphery of the world but was shortly to assume an important position as a maritime power.

So far as mathematics was concerned, Englishmen were not isolated from the scholars on the continent. Tartaglia had dedicated a book to Henry VIII in 1546. John Dee (1527–1608) studied on the continent and knew some of the outstanding mathematicians personally. Cardan visited London on his way to Scotland in 1552. Robert Recorde himself seems to have read the works of a number of these men, and we know that he made considerable use of Scheubel's algebra which was published in Paris in 1551.

But England had no mathematician to match the best of those in Europe, and so far as applied mathematics was concerned, the situation was deplorable. In England, navigation had not become a science. Map making was primitive. Surveying was casual. Little mathematics was taught. Arithmetic was neglected. Roger Ascham, tutor to the Princess Elizabeth, said that unless it was taught in moderation, mathematics overcharged the memory. It was a time when mathematics was needed to develop new techniques in surveying, navigation, and gunnery, but the workers knew no mathematics and the mathematicians, such as they were, lacked practical experience. The universities were of no help in the matter. The gap was bridged by a group, many of them amateurs, whom Dr. E. G. R. Taylor calls "Mathematical Practitioners."[3] Among them was Robert Recorde.

Recorde, a native of Pembrokeshire in Wales, studied at Oxford, received the degree of Doctor of Medicine from Cam-

[2] See Chapter 6 of G. M. Trevelyan, *History of England* (New York: Doubleday Anchor Books, 1953), Vol. II.

[3] E. G. R. Taylor, *Mathematical Practitioners of Tudor and Stuart England* (Cambridge, 1954).

bridge, and taught mathematics privately at both places, making the subject "clear to all capacities to an extent wholly unprecendented." He wrote a medical treatise of considerable importance, for it had at least ten editions from 1547 to 1665. He is reported to have practiced medicine in London, and it is claimed that he was physician to Edward VI and to Queen Mary. About 1551 he was appointed Surveyor of the Mines and Monies of Ireland and was made Comptroller of the Mint at Bristol. The *Whetstone of witte* ends abruptly with the entrance of a messenger sent to fetch the author to answer charges presumably of mismanagement of one of these political jobs. Within a year he died in the King's Bench Prison.

III. Recorde's mathematical books

Recorde was an opportunist. In his mathematical books, his aim was to provide the reader with needed information in palatable form. Recorde's four published books in mathematics were:

The Grovnd of Artes, an arithmetic, 1542 or perhaps 1540
The Pathewaie to Knowledge: First Principles of Geometry, 1551
The Castle of Knowledge, an astronomy, 1551
The Whetstone of witte, an algebra, 1557

In the prefaces of these volumes he refers to others he intends to write, in one case speaking of "other sundrye woorkes partely ended and partely to bee ended."

He took a realistic view of his qualifications as a writer. In the Preface of the *Grovnd of Artes*, he says

> . . . I know that no man can satisfie euery man, and therefore like as many do esteeme greatly other bookes, so I doubt not but some will like this my booke aboue any other English Arithmeticke hitherto written, & namely such as shal lacke instructers, for whose sake I haue plainly set forth the exa[m]ples, as no book (that I haue seene) hath hitherto.

His books were in the form of a dialogue between a Master and a Scholar, with the Scholar asking the leading questions in just the proper places. The books were well adapted to "such as shal lacke instructers."

The *Pathewaie to Knowledge* was dedicated to Edward VI with this explanation or apologia,

> Excuse me, Gentle Reader, if ought bee amisse' straunge pathes are not trode[n] al truly at the first: the way muste needes be comberous, wher none hathe gone before. . . . For neithe is my witte so finely filed, neither my learnyng so largely lettered neither yet my laisure so quiet and vncombered, that I maie performe iustely so learned a labour. . . . Yet may I thinke thus: This candle did I light: this light haue I kindeled: . . . I drew the platte [plan] rudelie wheron they maye builde, whom God hath indued with learnyng. . . . And this Gentle Reader I hartelie protest, where erroure hath happened I wishe it redreste.

As for the *Whetstone of witte*, Recorde is modest, but he felt that an algebra should be written and that it behooved him to do it. He says

> For better is it that a simple Coke doe prepare thy brekefast then that thou shouldest goe a hungered to bedde.

The *Grovnd of Artes* was a practical arithmetic. It found great use, as is witnessed by the fact that it had many editions over a period of a hundred and fifty years. It was quite in keeping with Recorde's purpose that the editors of later editions incorporated new things in them.

The *Pathewaie to Knowledge*, dedicated to Edward VI, was an introduction to geometry with definitions and conclusions in the first part, and proofs of the conclusions in the second. Third and fourth parts giving applications of these principles were promised but were never printed.

Recorde's astronomy, the *Castle of Knowledge*, dedicated to the Princess Mary, was both practical and theoretical. Dr. Taylor states that, "Like most physicians of his day, Recorde believed that the aspects of the heavens determined the correct times for taking medicines and other remedies, and he emphasized this equally with the needs of navigation for the study of astronomy." It has been claimed that Recorde's astronomy set forth the Copernican hypothesis, but Dr.

Taylor says that Recorde "expressed himself cautiously . . . but held no brief for an unmoveable earth circled by the stars and planets." Recorde might well be excused for being cautious for it must be remembered that the date of Copernicus's great work was 1543, and that this had a preface explaining that the hypothesis that the earth moved round the sun was merely a convenience in computation. It was after Recorde's time that Kepler detected that this was an interpolation.

IV. Recorde's algebra

The full title of Recorde's algebra is

The whetstone
of witte
whiche is the seconde parte of
Arithmetike: containyng the extrac
tion of Rootes: the *Cossike* practise,
with the rule of Equation: and
the woorkes of *Surde*
Nombers

Augustus de Morgan shows that this title is a play on words. The word *cos*, meaning a thing, was derived from the Latin *causa* by way of the Italian *cosa*. German and English writers used it to represent an unknown quantity. Consequently "Cossike practise" meant algebra. In Latin, the word *cos* means a grindstone. "Cossike practise" might be put into Latin as *cos ingenii* which in turn could be translated into English as the *Whetstone of witte*.

The preface is dated November 11, 1557. It should be remembered that at that time, the new year in England began March 25. There was time to print the volume before the end of 1557.

The book is dedicated to the "Venturers into Muscovia." In his introduction, Recorde says that he will

> . . . shortly set forthe soche a booke of Nauigation as I dare saie shall partly satisfie and contente, not onely your expectation but also the desire of a greate nomber beside wherin I will not forgette specially to touche bothe the olde attempte for the Northlie Nauigators, and the later good aduenture.[4]

In writing this algebra, Recorde referred to the work of Johannes Scheubel (1494–1570), professor of mathematics in the University of Tübingen, whose *Algebrae compendiosa facilisque descriptio* was published in Paris in 1551. Recorde objects to Scheubel's classification of equations, substituting one which he considered much simpler. Having thus improved upon Scheubel, Recorde proceeded to use direct translations of Scheubel's verbal problems. In fact, not only was Recorde acquainted with Scheubel's work but he plagiarized a considerable part of it. This is true not only in the section on algebraic equations, but also in the work on surds.[5]

V. The extraction of rootes

The first part of the *Whetstone of witte* considers numbers: whole numbers and broken numbers, "abstracte" and "contracte," evenly even and evenly odd. These last were of the type 2^n and $2(2^n+1)$. The ratios of numbers are classified with special names for each type. The subject of diametral numbers is treated in detail. A diametral number is the product of two integers which have the property that the sum of their squares is itself a perfect square. In other words, when a, b, and c are integers such that $a^2+b^2=c^2$, then ab is a diametral number. Recorde

[4] The reference here is to expeditions sent out by the "mystery and Company of Merchant Venturers" with the co-operation of Sebastian Cabot, newly returned from Spain. The purpose was to find a route to India and to wider markets for English products. The 'olde attempte' was an expedition to find the North West Passage. The 'later good aduenture' was sent out in 1553 to find a North East passage to India. It reached Russia instead. By penetrating the White Sea to the place where the port of Archangel now stands, the leader went overland to Moscow, where he met the tsar and obtained trading concessions. He thus circumvented the Hanseatic League which held a monopoly of trade through the Baltic, and opened the way for later commercial agreements between Russia and England. The backers of this expedition received a charter for trade with Russia and became the Moscovy Company. Robert Recorde was one of their advisers, presumably in questions of navigation.

[5] See Mary S. Day, *Scheubel as an Algebraist* (Contributions to Education No. 219 [Teachers College, Columbia University, 1926]).

shows that diametral numbers must end in 0, 2, or 8. They must be divisible by 12 and they cannot themselves be square numbers.

Recorde devoted over fifty pages to the extraction of roots. Here he gives particular attention to what he calls "lower ma[t]ters in warre." For example,

A citie should bee scaled, beyng double diched. And the inner diche .32. foote broade. And the walle .21. foote high. The capitain commaundeth ladders to be made of that iuste lengthe, that maie reche from the utter brow of the inner diche, to the toppe of the wal.

A bollette of yron of .7. inches diameter, doeth waie .27. pounds weighte: what shall be the diameter to that bollette that shall waie .125. pounde to the weighte?

VI. The cossike practice

The second part of the *Whetstone of witte* deals with algebraic numbers and with equations.

Following Scheubel's example, Recorde uses the symbol ℚ to denote a number, ℨ for the unknown quantity or root, ℨ for its square, and ℭ for its cube. The fourth order of these numbers was indicated by repeating the symbol for a square, i.e., a square of a square. Recorde's predecessors had carried this system to great lengths, but Recorde outdid them by taking it to the eightieth order. It should be noted that in any equation every term, even the constant one, had its symbol. Like Scheubel, Recorde also made use of abbreviations for these quantities—N, Ra., Pri., Se., Ter., and so on (the number, a root, the first product, the second product, etc). As an aid to the reader, Recorde lists the symbols in order, numbering them 1, 2, 3, . . and says "By this table, maie you easily knowe the signe that shall serve for your newe somme in multiplication."

The Scholar had difficulty in grasping the fact that .12.ℨ multiplied by .6.ℨ makes .72.ℭ. He says

This passeth my cunnynge, for the findyng of the newe signe although the multiplication of the nombers be as easie as can be.

The Master answers

If you did well remēber what you haue learned before; the māter would not seme so harde.

In dealing with "Cossike numbers," Robert Recorde introduced two signs, familiar to German writers, and familiar also to readers of the *Grownd of Artes*. He says

. . . touchyng these twoo signes + and − which bee the figures of more and lesse, you must giue regard whether thei bee like or unlike, in those numbers that must be added: For if thei be like in nombers of one denomination, then muste thei so remain as thei be. But if thei be vnlike, euermore abate the smaller nomber of theim that followe those unlike signes out of the greater, and sette doune the reste with the signe of the greater nomber.

The Scholar is then confronted with eight examples in which cossike numbers are to be added. These are the different arrangements of two basic problems:[6]

$$\begin{array}{rr} 10x \pm 12 & 10x \pm 8 \\ \underline{4x \pm 8} & \underline{4x \pm 12} \end{array}$$

He says

Here haue I varied one example diuersely, to the intente you maie marke the vse of your rules in theim.

He explains the addition of $4x-8$ to $10x+12$ as follows:

this sum is not fully $4x$ but wanteth of it 8 and therefore if you put downe $4x$ fully, you must abate 8 out of the 12 in the larger summe.

Recorde had a liking for putting things that were to be memorized into rhyme. The results were interesting. Here is his verse for the rules of signs in multiplication and division.

> who that will multiplie
> or yet divide trulie:
> shall like stille to haue more
> and mislike lesse in store.

The Scholar summarized it in these words:

So meane you that like signes multiplied together, doe make more, or + and vnlike sines multiplied together doe yelde lesse or − ?

In Recorde's work with polynomials, no powers of the cossike number are omitted.

[6] For convenience, modern symbols are used in these illustrations and in the balance of this paper.

For example, $8x^3+0x^2+0x+64$ is to be divided by $2x+4$.

Having practiced the fundamental operations with polynomials, the Master shows how to add fractions whose numerators and denominators are algebraic quantities. By his scheme, horizontal lines are drawn above and below the fractions that are to be added. The common denominator is written below the bottom line and the new numerators are written above the top line.

VII. "The Rule of Equation commonly called Algebers Rule"

This Rule is called the Rule of *Algeber*, after the name of the inuentoure, as some men thinke . . . but of his vse it is rightly called the rule of *equation:* bicause that by the *equation* of nombers, it doeth dissolue doubtful questions: and vnfolde intricate ridles.

Recorde's statement about the use of equations in solving problems is involved but worth considering.

When any question is propounded . . . you shall imagin a name for the nomber, that is to bee soughte, as you remember that you learned in the rule of false position. And with that nomber shall you procede, accordyng to the question, vntil you find a Cossike nomber equalle to that nomber that the question expresseth, whiche you shal reduce euer more to the leaste nombers.

The Scholar compares this with the rule of false, the method of solving problems by guessing the answer and then adjusting the guess until it fits the problem. The Master goes on

. . . it mai bee thoughte to bee a rule of wonderful inuention that teacheth a manne at the firste worde to name a true nomber before he knoweth resolutely [surely] what he hath named. But bicause that name is common to many nombers [although nct in one question] and therefore the name is obscure till the worke doe detect it, I thinke this rule might well bee called the rule of darke position, or of strange position, but not of false position.

And for the more easie and apte worke in this arte wee doe commonly name that dark position .1.ȝe and with it doe we worke as the question intendeth till we come to the equation.

At this point Recorde introduced the sign of equality, following it at once, as is

Howbeit, for eaſie alteratiō of *equations*. I will propounde a fewe exāples, bicauſe the extraction of their rootes, maie the more aptly bee wrought. And to auoide the tediouſe repetition of theſe woordes: is equalle to: I will ſette as I doe often in woorke vſe, a paire of paralleles, or Gemowe lines of one lengthe, thus: ═══, bicauſe noe. 2. thynges, can be moare equalle. And now marke theſe nombers.

14.ȝe. —+— .15.ꝙ ═══ 71.ꝙ.

20.ȝe. ——— .18.ꝙ ═══ .102.ꝙ.

26.ȝ. —+— 10ȝe ═══ 9.ȝ —— 10ȝe —+— 213.ꝙ.

Figure 1

shown in the accompanying facsimile, by a series of equations which he then considers in detail. In the case of $14x+15=71$, he says

. . . you mai see one denomination on both sides of the *equation* which neuer ought to stand. Wherfore abating the lesser, that is 15, out of both nombers, there will remain

$$14x=56$$

this is by reduction $x=4$ according to the third common sentence in the *Pathewaie*—If you abate euen portions from thynges that bee equalle, the partes that remain shall be equall also.

Other "common sentences" are used in solving other equations.

When confronted with the equation $26x^2+10x=9x^2-10x+213$, the Scholar wonders whether to add $10x$ or to "abate them." The Master says

In soche a case, you maie dooe either of bothe at your libertie and all will be to one ende. . . . And euermore when occasion serueth to translate nombers compounde, — on the one side is equalle to + on the other side.

Robert Recorde classifies equations into two groups:

1. Where one number is equal to another number,
2. Where one number is equal to two other numbers.

Here he has departed from Scheubel whose detailed classification can be represented as follows:

First type $bx=N$

Second type (1) $ax^2+bx=N$
(2) $x+N=x^2$ or
$bx+N=x^2$
(3) $bx+N=ax^2$
Third type $x^2+N=bx$

Neither writer classified equations according to their degree. Recorde includes the equation $6x^2=24x$ under his first heading, one number equal to another number, but he shows that the answer is not 4 but the square root of 4. He did not recognize roots that were zero or negative.

No reasons or explanations are given. Quadratic equations are solved by completing the square.

When the Master has completed the solution of a number of equations, the Scholar says "I doe couette some apte questions, appertainyng to these equations." The Master supplies them. They are not easy.

In one of these problems two men have silk to sell, one man has 40 ells, the other 90. The first man's silk is of the poorer quality, so he has to sell a third of an ell more for an angel than does the second man. If their total receipts are 42 angels, how many ells of silk did each sell for an angel? The Scholar makes a false start. He lets his unknown quantity represent the first man's receipts. The Master does not approve and suggests that he divide each man's ells by the number he sold for an angel and the quotient will be the man's receipts. Working this way, the Scholar finally comes to the conclusion that one man sold three ells for an angel and the other three and one third, so their receipts were 12 angels and 30 angels. Not content with this, the Scholar reverses the problem. Instead of using x and $x+\frac{1}{3}$, he uses $x-\frac{1}{3}$ and x. This time he gets one root correctly, but he also finds the other one, namely 2/21, and says

But how I maie frame that roote to agree to this question, I doe not see.

The Master is no help to him, except to say that

. . . the forme of the question maie easily instruct you whiche of these .2. rootes you shall take for your purpose.

As an example of a question where both roots fit the problem, the Master gives the following:

A gentilman, willyng to proue the cunnyng of a braggyng *Arithmetician*, saied thus: I haue in bothe my handes .8. crounes: But and if I accoumpte the somme of eche hande by it self seuerally and put therto the squares and the cubes of bothe, it will make in nomber 194. Now tell me (quod he) what is in eche hande: and I will giue you all for your laboure.

As would be expected, the answers 3 and 5 are the numbers in the man's hands.

In another case, the problem states that if 8 is added to a number and if 16 is taken from the square of the number, the product of the two results is 2560. This yields the equation $x^3+8x^2-16x=2688$. The Scholar is baffled. It is above his cunning, for here two numbers are equal to two others, or one number is equal to three numbers. The Master does not solve the problem. One suspects that he concocted the question by starting with the result. He simply says that the required number is 12 and has the Scholar show that this answer fits the equation. At this point the Master says

But to put you out of doubte, this equation is but a trifle to others that bee untouched.

There is no lack of variety in the problems. A man travels $1\frac{1}{2}$ miles the first day and increases his day's journey by $\frac{1}{8}$ mile each day. How long does it take him to go 2955 miles? In another case, the day's mileage increases in a geometric progression.

A herald is offered a bribe if he will tell the number of his king's army. The problem goes on this way—

The Heraulte lothe to lease those giftes, and as lothe to bee vntrue to his Prince, diuiseth his answere, whiche was true, but yet not so plain, that the aduersarie could thereby vnderstand that whiche he desired. And that aunswere was this. Looke how many Dukes there are, and for eche of them, there are twise so many Erles. And vnder euery Erle, there are fower tymes so many souldiars, as there be Dukes in the field. And when the muster of the soldiers was taken, the .200. parte of them was .9. tymes so many as the

nomber of the Dukes.
That is the true declaratio of eche number, quod the Heraulte: and I haue discharged my othe. Now guesse you how many of eche sorte there was.

A man dies leaving 72 crowns to his four children in this way: the second and third together were to have seven times as much as the first. The third and fourth were to have five times as much as the second. The first and fourth were to have twice as much as the third. It might be well to note that the children received the following sums: $4\frac{1}{2}$, $11\frac{1}{4}$, $20\frac{1}{4}$, 36.

A captain marshals his army in a square formation. When the square was of one size, he had 284 men too many, so he tried to rearrange them in a square one man more on a side than before. This time he lacked 25 men. How many men did he have?

VIII. The woorkes of surde nombers

The third part of the *Whetstone of witte* is devoted to irrational numbers, which Recorde calls surds. He says

Nombers *radicalle*, which commonly bee called nombers *irrationalle:* bicause many of them are soche, as can not bee expressed, by common nombers *abstracte*, nother by any certain rationalle nomber. Other men call them more aptly *surde* nombers. . . . A *surde nomber* is nothyng els, but soche a nomber set for a roote, as can not be expressed by any nomber absolute, as $\sqrt{10}$ or $\sqrt{18}$ or any nomber, that is not a square.

Surds are commensurable if they can be expressed as multiples of the same root; otherwise they are incommensurable.

Binomials made up of a rational number and an irrational one, are classified in two groups: "Nombers that be compounded with + be called Bimedialles and with − Residualles."

Accordingly, to divide by a binomial, dividend and divisor must be multiplied by the residualle of the divisor if the divisor is a bimedialle, or by the bimedialle of the divisor if the divisor be a residualle.

Recorde's square root sign was much like ours, but his signs for the cube root and the fourth root were perplexing. These are the symbols used by Scheubel, which he in turn had taken from the work of Rudolff (1525). When confronted with the symbol .⋀⋁. for cube root and .⋀⋁. for the fourth root, the Scholar speaks his mind.

It were againste reason, to take reason for these signes, which be set voluntarily to signifie any thyng; although some tymes there bee a certain apte conformitie in soche thynges. And in these figures, the nomber of their minomes [i.e. upstrokes] seameth disagreable to their order.

The Master replies:

In that there is some reason to bee thewed [instructed]: for as .√. declareth the multiplication of a nomber, ones by itself; so .⋀⋁. representeth that multiplication *Cubike*, in whiche the roote is represented thrise. And .⋀⋁. standeth for .√.√. that is .2. figures of Square multiplication: and is not expressed with .4. minomes. For so should it seme to expresse moare then .2. *Square* multiplications. But voluntarie [i.e., arbitrary] signes, it is inough to knowe that this doe signifie. And if any Manne can diuise other, moare easier or apter in use, that mai well be received.

IX. Appraisal of the Whetstone of witte

The chances are that the *Whetstone of witte* had few if any readers on the continent. Anyone interested in algebra would have found more satisfaction in Scheubel's concisely worded, elegantly printed treatise in Latin than in Recorde's popularization of it in English. So far as the equality sign was concerned, it is more than likely that Recorde had no particular pride in his invention. It was a thing he had found convenient. On the other hand, his simplification of Scheubel's classification of equations, in which he took apparent satisfaction, is of little interest today.

The *Grovnd of Artes* was closely connected with practical affairs. The *Whetstone of witte* seems to contain no practical applications, the reason being that algebraic problems which we would solve by equations of the first degree were treated by the rule of false in Recorde's arithmetic. The *Whetstone of witte* certainly had a smaller public. Had Recorde been able to complete the books that were "partely to bee ended," the algebra might

have had a wider circulation. As it was, the last of Recorde's books was the only one to have but a single edition.

Frequently the introduction of a book is best read after reading the book itself. So here, it seems appropriate to quote as a conclusion the verses on Recorde's title page:

Though many stones doe beare greate price,
The whetstone is for exercise
As neadfulle, and in woorke as straunge:
Dulle thinges and harde it will so chaunge,
And make them sharpe, to right good vse:
All artesmen know, thei can not chuse,
But vse his helpe: yet as men see,
Noe sharpnesse semeth in it to bee.
The *grounde of artes* did brede this stone:
His Vse is greate, and moare then one.
Here if you list your wittes to whette,
Moche sharpnesse therby shal you gette.
Dull wittes hereby doe greately mende,
Sharp wittes are fined to their full ende
Now proue, and praise, as you doe find,
And to yourself be not vnkinde.

Letters to the editor

Dear Sir:

I should like to describe the mathematics offering of the Bronx High School of Science somewhat more accurately and adequately than it was portrayed in Mr. Baumgartner's article, "The Status of Secondary Mathematics Program for the Talented" as found in the November (1956) issue of THE MATHEMATICS TEACHER.

Our Ninth, Tenth and Eleventh Year Mathematics courses are based upon the New York State Syllabus (obtainable from the State Education Department, Albany 1, New York). These courses include Elementary Algebra, Plane Geometry, Co-ordinate Geometry, Intermediate Algebra, and Trigonometry. In this school, all the mathematics courses are considerably enriched and extended for all classes. Topics such as determinants, advanced factoring, inequalities, etc., are in our "regular" course.

Our "regular" elective program includes the following courses:

1. Advanced Algebra followed by a Higher Geometry course,
2. Advanced Algebra followed by a Higher Algebra course,
3. A combination of 1 and 2,
4. A one-year course in the mathematics of science, for which either 1 or 2 is corequisite,
5. Advanced Algebra followed or preceded by a course offered by the Science Departments, "The History and Development of Science."

The Advanced Algebra course includes a substantial introduction to the calculus.

The Higher Geometry course, which replaced Solid Geometry, includes a unit on the postulational basis of solid geometry, a unit on solid analytics, and a unit on vector analysis. The unit on vector analysis, developed by Mr. A. Glicksman, is a rigorous postulational system based upon algebra rather than geometry.

The Higher Algebra course is based upon Allendoerfer and Oakley's *Principles of Mathematics*. It includes the theory of groups, rings and fields, some function theory, and additional work in the calculus. This course was developed by Messrs. Ruderman and Glicksman.

The Mathematics of Science course uses ten reference books, plus mimeographed material. The units include navigation, surveying, statistical analysis, biometrics, physical chemistry, theoretical physics and higher mathematics. This course was prepared by Mr. Greitzer.

At the present time, the Mathematics Department is working on a new Eleventh Year course of study involving a systematic development of algebra from group and field theory. Experimentation will take place in the 1957-1958 school year.

In addition to the "regular" mathematics program, described above, we have an "honors" mathematics program. Students are allowed to apply for this accelerated program at the end of the tenth year. Of the applicants, approximately two classes are selected. The minimum requirement is 90% in every major, no difficulty of any kind, excellent achievement and interest in mathematics, and the recommendation of the teachers. In the eleventh year, these two classes take a combined course including Eleventh Year Mathematics and almost all of Advanced Algebra, with all optional topics recommended in the Kenyon Plan. During the summer, after the end of the eleventh year, students are required to learn Solid Geometry by home study. Upon returning to school, these students undergo an examination which emphasizes that part of Solid Geometry which is indispensable in the study of the Calculus. There are no proofs in this examination.

The successful candidates enter the "College Mathematics" class in the twelfth year. Here they study Analytic Geometry and the Calculus. The textbooks are Wilson and Tracey,

Continued on page 279

On rapid sketching of polar curves

DONALD GREENSPAN, *Hughes Aircraft Company, Culver City, California.*

Here is an interesting technique whereby polar curves are drawn with the aid of an r-θ rectangular co-ordinate sketch.

I. Introduction

One of the most important heuristic devices which engineers and mathematicians can possess is a rapid, accurate method for curve sketching, since the "picture" of a function is often more enlightening than its analytic representation. The plebian technique of plotting points to effect the graph of a polar curve is exceedingly laborious, often fruitless, and rather dull. Below is proposed a method which with practice will usually require from only three to fifteen seconds to yield an accurate graph. The difficulty in describing the procedure in words far exceeds in difficulty the procedure itself. Pedagogical experience has shown that at least twenty minutes of a class period should be devoted to practice problems which a student graphs for himself.

II. Preparatory material

It is necessary first to recall the well-known relationships connecting polar and rectangular co-ordinates, i.e., $x=r\cos\theta$, $y=r\sin\theta$; $x^2+y^2=r^2$; $\theta=\tan^{-1}y/x$. Plotting the following points will demonstrate some of the subtler aspects of polar co-ordinates:

A: $r=2$, $\quad\theta=30°$

B: $r=2$, $\quad\theta=\dfrac{\pi}{3}$

C: $r=-2$, $\quad\theta=30°$

D: $r=-2$, $\quad\theta=90°$

E: $r=0$, $\quad\theta=0°$

F: $r=0$, $\quad\theta=30°$

G: $r=2$, $\quad\theta=315°$

H: $r=-2$, $\quad\theta=315°$

I: $r=2$, $\quad\theta=690°$

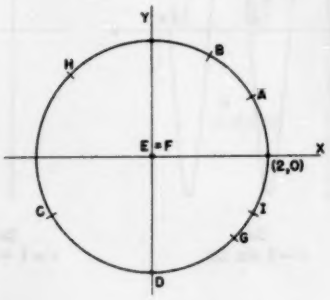


Figure 1

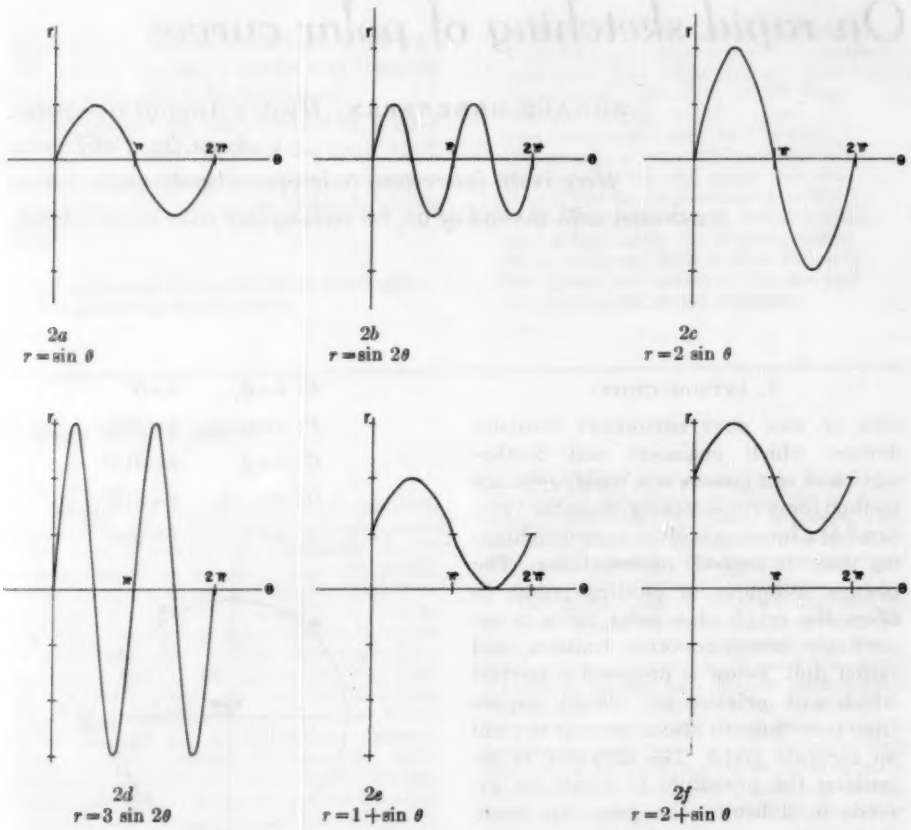
It is then necessary to review and/or become facile with some, perhaps new, trigonometric graphs. A possible order of discussion is presented below for functions of $\sin\theta$. An analogous treatment could or should be presented for $\cos\theta$.

Concerning the graphs in Figure 2*a* to 2*l*, the following should be noted:

(a) In each case $r=f(\theta)$ is graphed in an $r-\theta$ co-ordinate system.

(b) If $r=f(\theta)$ is graphed, then $r=-f(\theta)$ is the symmetric image about the θ axis (compare Figures 2*a* and 2*h*).

(c) An expression like: $r=1-3\sin 2\theta$, can be graphed in steps by considering: $3\sin 2\theta$, then: $-3\sin 2\theta$, then: $1-3\sin 2\theta = 1+(-3\sin 2\theta)$.



2a $r=\sin\theta$ 2b $r=\sin 2\theta$ 2c $r=2\sin\theta$

2d $r=3\sin 2\theta$ 2e $r=1+\sin\theta$ 2f $r=2+\sin\theta$

Figure 2

III. The method

The problem is to sketch the curve: $r=f(\theta)$, in the $X-Y$ co-ordinate system. In general, the procedure is to first sketch an auxiliary graph: $r=f(\theta)$, in an $r-\theta$ rectangular system, as in the preceding section. This $r-\theta$ graph yields quite clearly the variation of r with θ. For example, in (2*e*), one easily sees that as θ increases from 0 to $\pi/2$, r increases from 1 to 2, and as θ increases from $\pi/2$ to π, then r decreases from 2 to 1, etc.

It is this variation which is of the essence for the sketch in the $X-Y$ plane. For, in the $X-Y$ plane, θ is the angle, measured counterclockwise, which the vector OQ (see Figure 3a), of length r, makes with the positive half of the X-axis. One may think of vector OQ rotating counterclockwise, but changing in length as it rotates. This variation of length depends on θ and is to be determined from the auxiliary graph. Hence, the point Q traces out the desired graph.

Example 1: $r=1+\sin\theta$.

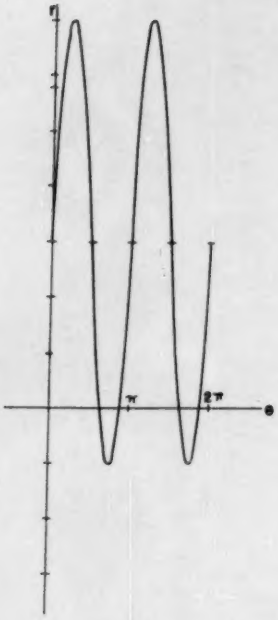
Steps:

(a) Sketch the auxiliary graph. (See Figure 2*e*.)

(b) Note from Figure 2*e* that $r=1$ when $\theta=0$, so that $Q(1, 0)$ is a starting point in the $X-Y$ system. (See Figure 3*b*.)

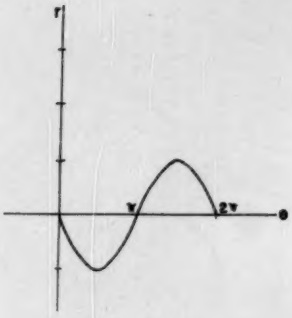
(c) From Figure 2*e*, it follows that as θ goes from 0 to 90°, r increases from 1 to 2. This yields Figure 3*c*.

(d) From Figure 2*e*, it follows that as



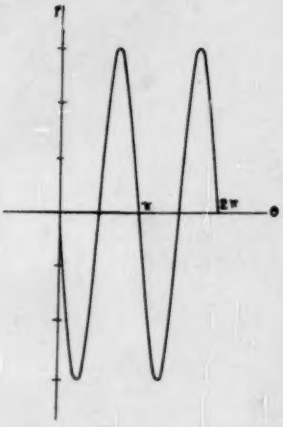
2g

$r = 3 + 4 \sin 2\theta$



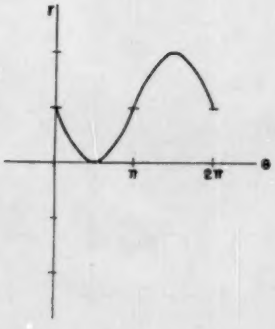
2h

$r = -\sin \theta$



2i

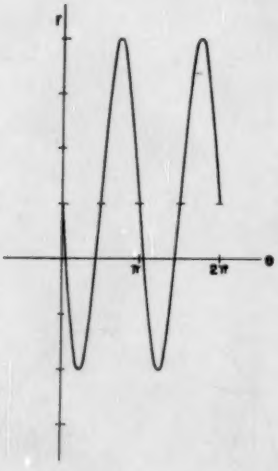
$r = -3 \sin 2\theta$



2j

$r = 1 - \sin \theta$

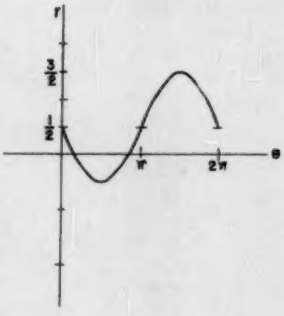
$= 1 + (-\sin \theta)$



2k

$r = 1 - 3 \sin 2\theta$

$= 1 + (-3 \sin 2\theta)$



2l

$r = \dfrac{1 - 2 \sin \theta}{2}$

$= \frac{1}{2} + (-\sin \theta)$

Figure 2 (continued)

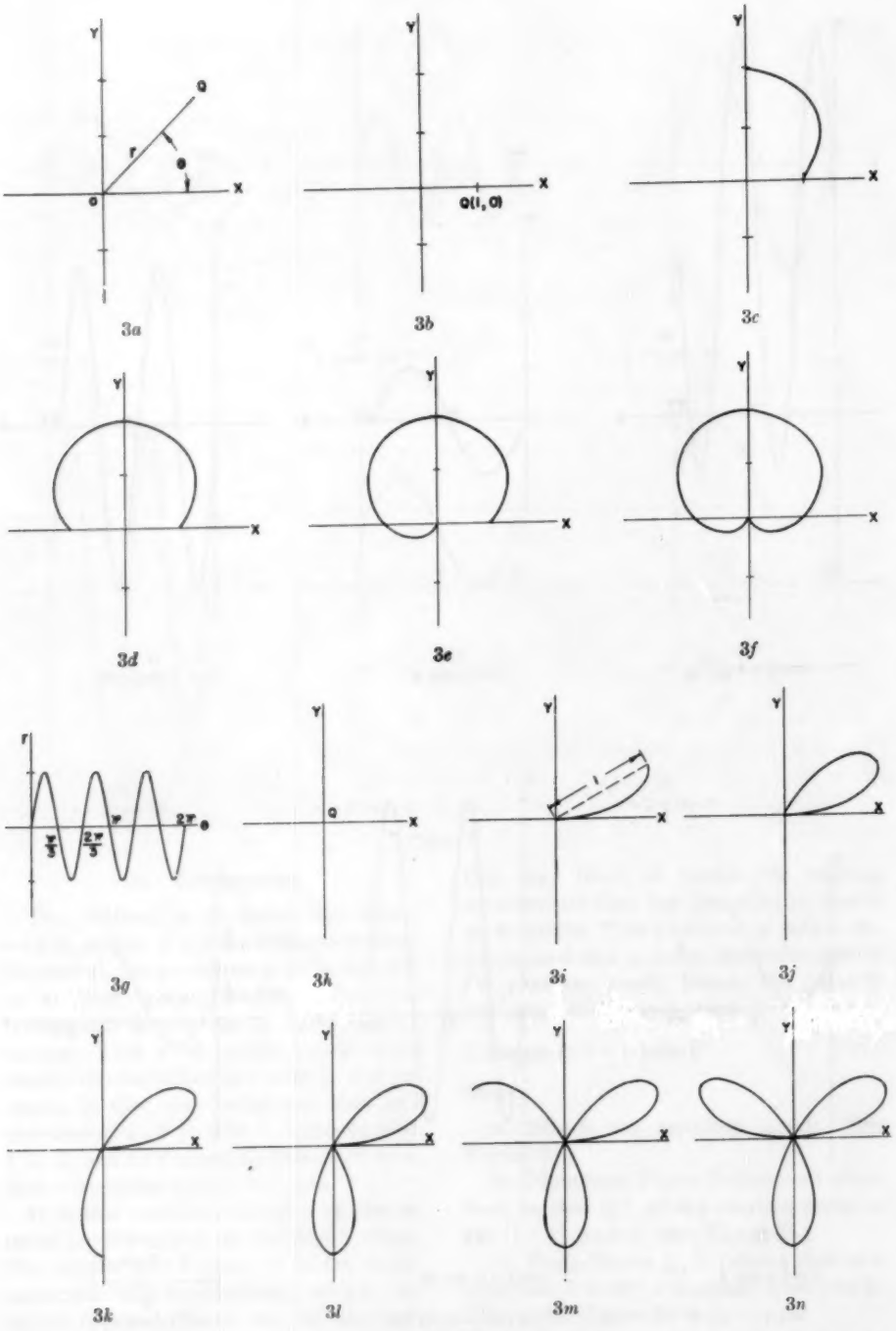
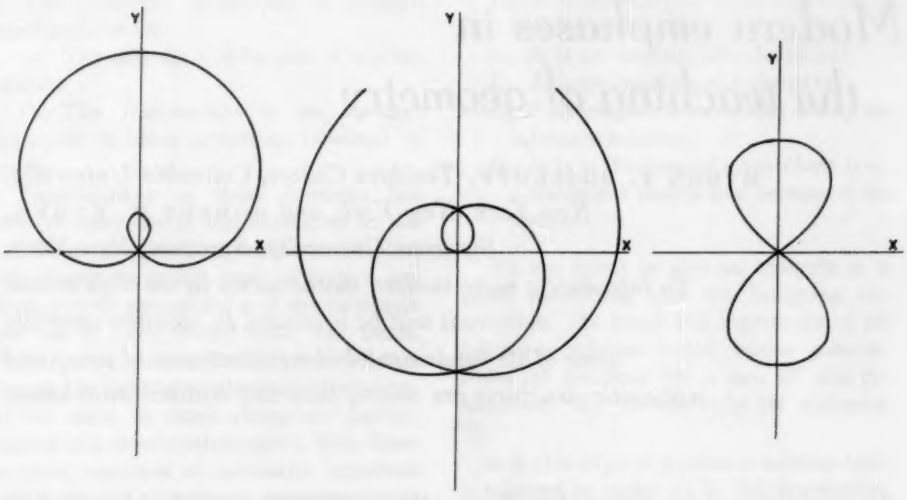


Figure 3



4a 4b 4c

Figure 4

θ goes from 90° to 180°, r decreases from 2 to 1. This yields Figure 3*d*.

(e) From Figure 2*e*, it follows that as θ goes from 180° to 270°, r decreases from 1 to 0. This yields Figure 3*e*.

(f) From Figure 2*e*, it follows that as θ goes from 270° to 360°, r increases from 0 to 1. This yields Figure 3*f* and the graph.

Note that only one point was actually plotted and this was the starting point (1, 0).

Example 2: $r=\sin 3\theta$

Steps:

(a) Sketch auxiliary graph. (See Figure 3*g*.)

(b) Note from Figure 3*g* that $r=0$ when $\theta=0$ so that Q (0, 0) is a starting point in the $X-Y$ system. (See Figure 3*h*.)

(c) From Figure 3*g* it follows that as θ varies from 0 to 30°, r varies from 0 to 1. This yields Figure 3*i*.

(d) From Figure 3*g* it follows that as θ varies from 30° to 60°, r decreases from 1 to 0. This yields Figure 3*j*.

(e) From Figure (3*g*) it follows that as θ varies from 60° to 90°, r decreases from 0 to -1. This yields Figure 3*k*.

(f) From Figure 3*g* it follows that as θ varies from 90° to 120°, r increases from -1 to 0. This yields Figure 3*l*.

(g) From Figure 3*g* it follows that as θ increases from 120° to 150°, r increases from 0 to 1. This yields Figure 3*m*.

(h) From Figure 3*g*, it follows that as θ increases from 150° to 180°, r decreases from 1 to 0. This yields Figure 3*n*, which is the desired graph.

IV. Concluding remarks

In practice, there is of course no need for the multitude of $X-Y$ graphs given in the examples nor for a written description of the variation of r with θ.

Other interesting functions whose graphs are easily constructed by this method are:

(a) $r=2+3\sin\theta$
(b) $r=\theta$
(c) $r=\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$.

Their graphs are, respectively, shown in Figure 4.

Finally, it may be noted that there is a somewhat similar technique for the rapid graphing of curves given parametrically. But this is another story.

Modern emphases in the teaching of geometry

MYRON F. ROSSKOPF, *Teachers College, Columbia University, New York, New York, and* ROBERT M. EXNER, *Syracuse University, Syracuse, New York.*

To introduce a more modern mathematics in the high school, it will be necessary to examine carefully some of its fundamental ideas. Strategies of proof and axiomatic structures are among these key mathematical ideas.

RECENTLY there have appeared in THE MATHEMATICS TEACHER and *School Science and Mathematics* a number of papers that consider trends in secondary mathematics. Perhaps the three most significant for the purposes of the present paper are the two by Wilson[1] and one by May.[2] These men indicate clearly what was done in the past as well as present trends.

It is no longer news that mathematics programs are in a state of ferment. All mathematics teachers are aware of changes in undergraduate mathematics and observe in new textbooks for high school students evidence of changes in secondary mathematics. Even greater changes will occur in the next few years if the efforts of leaders in mathematics education and of committees of a variety of organizations are fruitful. Motivation for change is found in the unique relationship between the traditional secondary mathematics program and the historical development of mathematics. It is a rare high school program that contains any mathematics not already well organized by the end of the seventeenth century. Among high school subjects, mathematics and ancient history are the only ones that do not come within 250 years of the present. If one compares the amount of mathematics developed by the end of the seventeenth century with the tremendous outpouring of mathematics since that time, it is not hard to find a strong flavor of ancient history in the traditional high school mathematics courses.

In casting about for ways of bridging the 250-year gap in mathematics for high school students, one is continually blocked by the sequential nature of the subject. The subject does not lend itself to condensation. Some time might be saved in algebra, for instance, by eliminating drill work with exponentials of the "spreading chestnut tree" type along with the "chinese pagoda" type of radicals. In the end, the time saved by all such eliminations would not allow the introduction of very much new mathematics. A more promising line might be to try to teach algebra and geometry in a more modern spirit. To discover this modern spirit requires studying contemporary mathematics to search out the concepts that are fundamental. A high school program can then lay the foundation for successive stages of development of these concepts.

[1] Jack D. Wilson, "Trends in Geometry." THE MATHEMATICS TEACHER, XLVI (February 1953), 67–70; "Trends in High School Mathematics." *School Science and Mathematics* LV (June 1955), 462–468.

[2] Kenneth O. May, "Which Way Pre-college Mathematics?" THE MATHEMATICS TEACHER, XLVII (May 1954), 303–307.

Two concepts important in modern mathematics are:

(a) The axiomatic structure of mathematics.

(b) The relationship of an abstract structure to other structures (abstract or real).

Development of these concepts has always been one of the objectives in the teaching of secondary mathematics, but the degree to which each objective has been clearly recognized and the emphasis put on it have varied with the times. Today, the concepts themselves are changing in the light of modern developments. If we want to teach secondary mathematics in a more modern spirit, then these modern concepts of axiomatic structure and relationships between structures must be reflected in our teaching.

Proof

Fundamental in a concept of axiomatic structure is a concept of *proof*. Particularly in geometry classes, teachers try to teach what is meant by *mathematical proof*. Thus we must ask, "Do we, as teachers, have a clear and modern conception of *mathematical proof*?"

In a previous paper[3] the authors gave a modification of a description of *demonstration* or *proof* as given by Rosser[4] and others. There follows a description of what might be called an *abbreviated proof:*

An abbreviated proof of a conclusion Q on the basis of certain hypotheses H_1, H_2, $\cdots$, H_n is a sequence of statements:

$$S_1, S_2, \cdots, S_k,$$

where S_k is the same as Q and where each S_i of the sequence is one of the following:

D_1: S_i is an instance of an axiom,
D_2: S_i is the same as some H_j,
D_3: S_i is the same as some earlier S_j in the sequence,
D_4: S_i is an instance of a definition,
D_5: S_i is an instance of a theorem,
D_6: S_i is inferred from earlier S's by an inference scheme,
D_7: S_i is in the form of a tautology (i.e., a statement that is true because of its form).

We are about to give an example of a proof consistent with the foregoing description. The proof will require use of an inference scheme called *modus ponens: From the statement "If* A *then* B" *and the statement* "A," *we can infer the statement* "B."

It is this type of inference scheme that is referred to under D_6 in the description of a proof. Modus ponens justifies including the statement "B" as a step in a proof if the statements "If A then B" and "A" have already occurred in the proof. In the example, given at the top of the facing page, those statements that result from use of modus ponens have in their corresponding analysis item an indication of the statements used in the inference. While you read this proof, note that it is the sequence of statements in the left-hand column that constitutes the proof. The analysis column merely indicates the sort of statement that is used by referring the reader to the previous description of a proof.

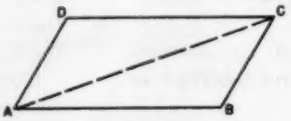
The resulting proof seems, at first glance, more lengthy than the proofs of this theorem that we are accustomed to, but in fact, we have included little in our proof that does not occur in conventional forms of proof. The difference lies in the organization into a column of statements and a column of analysis. Our analysis column is purely a commentary on the proof, and if it were omitted, the remaining column of statements would constitute a complete and intelligible argument. The usual organization into "Statements" and "Reasons" does not have this property; the "Reasons" column usually contains both comments on the proof and

[3] Myron F. Rosskopf and Robert M. Exner, "Some Concepts of Logic and Their Application in Elementary Mathematics." The Mathematics Teacher, XLVIII (May 1955), 290–298.

[4] J. Barkley Rosser, *Logic for Mathematicians* (New York: McGraw-Hill Book Company, 1953).

Theorem: If quadrilateral $ABCD$ has $AB=CD$ and $AB\|CD$, then quadrilateral $ABCD$ is a parallelogram.

Figure 1



Hypothesis: Quad. $ABCD$ with $AB=CD$ and $AB\|CD$.
Conclusion: Quad. $ABCD$ is a parallelogram.

Statement	*Analysis*
S_1: Quad. $ABCD$ with $AB=CD$ and $AB\|CD$.	D_2
S_2: On two distinct points A and C there is one and only one line AC. (AC forms with the sides of $ABCD$ the triangles ABC and ADC.)	D_1
S_3: If $AB\|CD$, and AC intersects AB and CD, then $\angle BAC=\angle DCA$.	D_5
S_4: $\therefore \angle BAC=\angle DCA$.	D_6: Mod. pon. (1, 2, 3)
S_5: $AC=AC$	D_1: Identity
S_6: If in triangles ABC and ADC, $AB=CD$, $\angle BAC=\angle DCA$, $AC=AC$, then $\triangle ABC\cong\triangle ADC$.	D_5
S_7: $\therefore \triangle ABC\cong\triangle ADC$.	D_6: Mod. pon. (1, 4, 5, 6)
S_8: Triangles ABC and ADC are congruent if and only if their corresponding parts are equal.	D_4
S_9: $\therefore \angle ACB=\angle CAD$.	D_6: Mod. pon. (7, 8)
S_{10}: If AD and BC intersect AC so that $\angle ACB=\angle CAB$, then $AD\|BC$.	D_5
S_{11}: $\therefore AD\|BC$.	D_6: Mod. pon. (9, 10)
S_{12}: Quad. $ABCD$ is a parallelogram if and only if $AB\|CD$ and $AD\|BC$.	D_4
S_{13}: $\therefore$ Quad. $ABCD$ is a parallelogram.	D_6: Mod. pon. (1, 11, 12)

statements that should be part of the proof. It is just this that makes it so hard to describe for the student what proper reasons are. In a conventional form of proof, our statements S_2, S_6, S_8, S_{10}, S_{12} would normally appear as reasons for S_4, S_7, S_9, S_{11}, S_{13}, respectively.

The writers are convinced that the average mathematics student rarely learns what a proof *is*, but rather learns *how* to write correct proofs through imitation of his instructor and his textbook and by adjusting his efforts to their authority. Whatever else, the student learns that for every statement there must be a reason, although he is vague about the relation between statement and reason, and vaguer still about the relation between statement and succeeding statement. The writers maintain that a student who can write such a proof as the foregoing understands both the facts and principles of geometry *and* something about the nature of proof.

There is more to the business of "formal proofs" than has been indicated here. We have tried to make the point that there is a great deal that needs to be done, and can be done, to give high school students a better concept of the nature of proof.

Axiomatic structure

Before a start can be made in an analysis of *axiomatic structure*, a description is

needed. A statement by Wilder[5] will serve to start us off.

> . . . the axiomatic method consists in setting forth certain basic statements about the concept (such as the geometry of the plane) to be studied, using certain undefined technical terms as well as the terms of classical logic.

It is well in such a discussion, to distinguish between the formal abstract axiomatic system itself, and the conceptual structure of which it is to be a formalization. For a formal axiomatic system, we presuppose a language that includes a logic giving rules of inference for deducing new statements from given statements. To this we add some undefined descriptive words (or symbols). In a geometry, among such words might be "point," "line," "between." In an abstract physical science, some undefined words might be "time," "mass," "matter." The next step is to set forth the axioms, which are statements using the undefined words. By means of the assumed logic, further statements called theorems, involving the undefined words, can be deduced. The entire structure consisting of the language, the logic, the undefined words, the axioms, and the theorems constitutes an abstract axiomatic system.

From this formal viewpoint, it makes no sense to talk about the truth of the axioms or theorems. The undefined words are meaningless, and statements involving them are equally meaningless. However, we can inject meanings by correlating the undefined words with words in some other system (either another abstract system or the real world). Then the axioms and theorems become statements in the other system, and are said to be *interpreted*. Thus the primitive Euclidean notion of "line segment" may be interpreted as "the edge of a board," and its "length" determined by comparing it with a standard yard. Of course, some of the interpreted theorems or axioms may be false statements in the other system, in which case the interpretation is called *invalid*. From the pragmatic point of view, the valid interpretations are of interest, and in what follows we shall use *interpretation* to mean *valid interpretation* and shall leave the word valid tacit.

As Wilder's description indicates, in building an axiomatic system one generally has in mind a specific interpretation for it. This ultimate interpretation guides the selection of undefined terms and axioms. However, an abstract axiomatic system may well have several interpretations essentially different from the one which motivated its construction.

Certainly Euclidean geometry has strong empirical roots. Indeed, as presented in the *Elements* it is not an abstract axiomatic system as outlined above, but is a mixture of abstraction and interpretation. It was only in comparatively recent times that mathematicians (see Hilbert[6] and Veblen[7]) succeeded in constructing a purely abstract axiomatization of Euclidean geometry. The systems of Veblen and Hilbert are too sophisticated for high school students who must learn logic along with the geometry. It seems clear that a good deal of empirical geometry will always be a part of the high school course. However, it seems to the authors that there should be more emphasis put on the distinction between abstract geometry and interpreted or empirical geometry in the high school course.

If axiomatic systems are considered apart from any specific interpretations, the question arises: Having chosen the undefined words, can you choose any arbitrary set of statements as the axioms of a system? The answer seems to be a

[5] Raymond L. Wilder, *Introduction to the Foundations of Mathematics* (New York: John Wiley and Sons, Inc., 1952), p. 9.

[6] David Hilbert, *Grundlagen der Geometrie*, Seventh Edition (Leipzig: Tenbuer, 1930) or *The Foundations of Geometry*, authorized translation of the first edition by E. J. Townsend (Chicago: Open Court Publishing Company, 1902).

[7] Oswald Veblen, "A System of Axioms for Geometry," *Transactions of the American Mathematical Society*, V (1904), 343–384.

qualified no. There are certain characteristics that mathematicians require of an axiomatic system. In fact, part of the study of axiomatic systems is devoted to determining whether a given system has these desirable characteristics.

In the case of the axioms, it is considered desirable that they be *consistent, independent,* and *complete.* Sometimes it is also required that they be *categorical.*

Let us illustrate some of these attributes by the following set of axioms for a *miniature geometry;* that is, a geometry in which there are a finite number of points and lines.

I. *Undefined elements:* "point," "line." (Notation: lower case letters will stand for lines; upper case letters for points.)

Undefined relation: "on." ("Point on line" and "line on point" will be interpreted to have the same meaning.)

II. *Axioms:*

A_1: Each pair of lines is on at least one point.

A_2: Each pair of lines is on not more than one point.

A_3: Each point is on at least two lines.

A_4: Each point is on not more than two lines.

A_5: The total number of lines is 4.

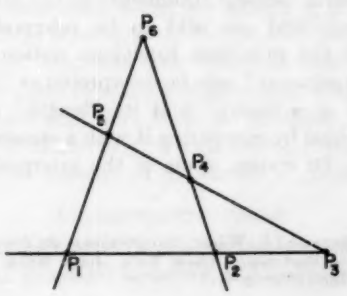
Let us call the system consisting of the undefined elements, undefined relation, and axioms, the *axiom system S.*

With a specific set of axioms before us, we can proceed to a discussion of the desirable properties of such a system. First, let us consider consistency.

An axiom system is called consistent if contradictory statements are not implied by it.[8] This definition of consistency of a set of axioms just given is good enough, but leaves open the question of just how consistency can be proved. Does the definition mean that we must derive all the theorems implied by the set of axioms and then test these theorems to see if any pair is contradictory? Such a task seems impossible of accomplishment. The difficulty is resolved—perhaps unsatisfactorily—by agreeing that a set of axioms is consistent if it can be interpreted in some other abstract system which is consistent. It is clear that if our set of axioms does imply contradictory statements A and B, then the interpretations of A and B will be contradictory statements in the other system, which is impossible since it is supposed to be consistent. This method of determining consistency depends upon finding a consistent system in which to make the interpretation. Unfortunately, it is not generally possible to find a consistent system of much complexity, so that usually the most one can say is that a given axiomatic system is as consistent as some other one. In the case of the Axiom System S, we will interpret it in Euclidean geometry. No one knows whether Euclidean geometry is consistent, but at least it has been around a long time and in its modern formulation no one has found any contradictions in it.

The interpretation is made by constructing in the Euclidean plane a model consisting of certain lines and points, which will be interpreted as the "points" and "lines" of the axiom system S. The undefined word "on" of S will be interpreted as the word "on" of Euclidean geometry. Then Figure 2 pictures a valid model, as can be verified by examining each of the axioms in turn.

Figure 2



[8] Raymond L. Wilder, *op. cit.*, p. 23 and Bruce E. Meserve, *Fundamental Concepts of Geometry.* (Cambridge, Massachusetts: Addison-Wesley Publishing Company, 1955).

Here we have given one interpretation of the axiom system S. Other interpretations exist. For example, by assigning proper meanings to the undefined elements and relations one could construct a model using (1) people and committees, (2) lines and planes, (3) colors and color schemes, (4) numbers and number pairs.

To say that the axioms of the set are independent means that no one of the axioms can be deduced from the others. A definition of independence that furnishes a method for showing independence can be made as follows, where we shall agree to designate the negation or denial of axiom A by "$\sim A$," and to designate the system obtained from S by deleting A by "$S-A$":[9]

If S is an axiom system and A is one of the axioms of S, then A is independent in S if there exists an interpretation for S and an interpretation for $(S-A)+(\sim A)$.

The idea here is that if A is deducible from the remaining axioms of S, then A is deducible from these same axioms in $(S-A)+(\sim A)$. But then $(S-A)+(\sim A)$ contains the contradictory statements A and $\sim A$, so that not both S and $(S-A)+(\sim A)$ can be consistent.

Since we already have an interpretation for our axiom system S, that point in the test for independence is taken care of. There remains the task of testing each axiom and providing an interpretation for $(S-A_i)+(\sim A_i)$, where $i=1, 2, 3, 4, 5$.

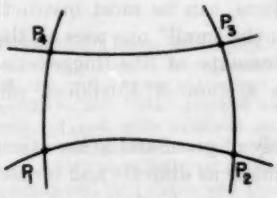


Figure 3

1. $(S-A_1)+(\sim A_1)$. Figure 3 is the desired interpretation or model. Lines P_1P_2 and P_3P_4, P_1P_4 and P_2P_3 have no point in common; yet the other axioms are satisfied.

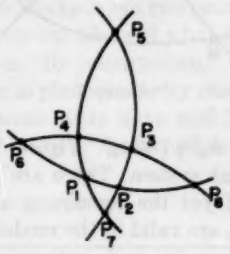


Figure 4

2. $(S-A_2)+(\sim A_2)$. In Figure 4 it is clear that each pair of lines is on at least one point; each point is on at least two lines; each point is on not more than two lines; the total number of lines is four; but there are two pairs of lines that are on two points.

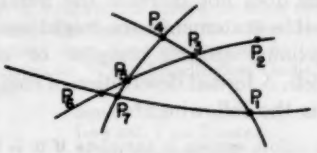


Figure 5

3. $(S-A_3)+(\sim A_3)$. Figure 5 is the interpretation for this situation. P_2 is on only one line.

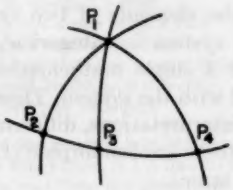


Figure 6

4. $(S-A_4)+(\sim A_4)$. In Figure 6, each pair of lines is on at least one point; each pair of lines is on not more than one point; each point is on at least two lines; the total number of lines is four; but there is one point, P_1, on more than two lines.

[9] Raymond L. Wilder, *op. cit.*, p. 29.

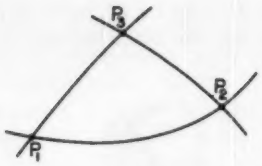


Figure 7

5. $(S-A_5)+(\sim A_5)$. Figure 7 satisfies the present system. There are just three lines, and yet the remaining axioms A_1, A_2, A_3, A_4 are valid in the model.

Completeness and categoricalness of axiom systems are aspects of formal investigations in studies of axiomatic systems. The notion of *completeness* is concerned with the size of the body of statements in the system that are implied by the axioms. Intuitively, one would expect that increasing the number of axioms of a system would increase the number of deducible statements. If such an increase in the axioms does not increase the number of deducible statements we might consider the axioms *already complete* or simply *complete*. A formal description of completeness is the following:[10]

> An axiom system is complete if it is impossible to state another axiom in terms of the undefined elements such that the new axiom is independent of the system.

On the other hand, categoricalness involves formulating a definition of *one-to-one correspondence* and an *isomorphism* between the elements of two systems. If an axiom system is categorical, there is essentially a single mathematical theory associated with the system. There may be different interpretations, different models, but all of these are isomorphic (i.e., essentially the same).

Which of these four properties, consistency, independence, completeness, categoricalness, is it essential that an axiom system possess? Well, all axiom systems ought to be consistent. It is very embarrassing to set up a system of axioms and find you are able to prove contradictory statements! What is worse, in such a system, any statement at all can be proved. As you know from your experience with teaching plane geometry, independence of axioms is not an essential characteristic of a system. All current geometry textbooks use a dependent set of axioms for pedagogical reasons. For example, many textbooks assume the three congruence theorems, some of which you know can be proved. Similarly, categoricalness and completeness have disadvantages and advantages. Noncategorical and incomplete systems can be very useful in mathematics precisely because they do have many different interpretations and can be extended. The axioms for a *group* constitute such a system in algebra.

[10] Raymond L. Wilder, *op. cit.*, p. 33.

Summary

In this paper we discussed two aspects of modern mathematics that need emphasis in the teaching of geometry: axiomatic structure, and interpretations of such structures. As a corollary, we feel that the nature of proof needs to be highlighted in teaching mathematics so that an idea of what a mathematical proof is becomes part of the conscious mathematical equipment of students. Axiomatic systems play such an important and fundamental role in mathematics today that it seems necessary to lay a foundation in high school for their further study. A consideration of a miniature geometry, like the one presented here, can be most instructive, for here "in the small" one sees all the structural elements of the bigger and very complex system of Euclidean plane geometry.

A study of proof and axiomatic systems can be made as difficult and involved as a teacher pleases. Certainly, it is not the intent of the authors that more than correct foundations be laid in high school classes. However, the authors insist it is possible to develop a *concept* of what a proof is with high school students as well as a *process* or form of a correct proof.

Some elements of the structure of geometry appear in existing plane geometry textbooks. Such elements of structure should appear more often in other mathematics courses also.

One aspect of an axiomatic system that seems not clearly understood is the distinction between an abstract system and an interpretation of it. The language we use might help students acquire a correct concept. For example, it would be well to state an axiom like "There *is* one and only one line on two distinct points" as an axiom of existence rather than as an axiom of construction: "One and only one line can be *drawn* between two points." We need to clarify our ideas of what is meant by the reason "By construction," which is so often used in plane geometry classrooms. Perhaps it would help if we said "Draw a *picture* of Line AB" rather than "Draw *the* line AB."

Letters to the editor

Continued from page 266

and Thomas. The work in Thomas' book is equivalent to m11, m12 and most of m21 at the Massachusetts Institute of Technology. Students in this course are also required to attend a Seminar for all honors students in which the Mathematics and Science Departments take joint responsibility for bringing these accelerated students up to date on new trends in the four fields involved.

The purpose of this letter is to show that the talented students at this school are benefiting from the best mathematics that can be taught. The Department of Mathematics is exceptionally able and professionally minded. The students are exceptionally capable and interested. Neither one nor the other is a sufficient condition for success, but the two, together, may explain the records our graduates have made.

Very truly yours,
IRVING ALLEN DODES
Chairman
Department of Mathematics

Dear Sir,

I am submitting a poem that a pupil of mine wrote in class one day. I had placed on the blackboard the well-known statement of Hilbert's on mathematics: "Mathematics is a meaningless game, played with symbols on paper, according to certain specified rules (or axioms)." My plane geometry class had just finished their work on axioms and postulates, and this led me to request a short paper, in way of comment on Hilbert's statement. To the best of my knowledge, no pupil had seen it before that time. All students except one wrote a composition, many decrying and taking violent exception especially to the "meaningless" aspect of the game. However, one student, Linda Chamberlin, composed, on the spur of the moment, the poem which I enclose. It was substantially in this form after the half hour's work. Since I thought it very original, I suggested that she "polish" it with her English teacher, Mr. Claude T. Burns, also here at Herbert Hoover High School. If there is space available, we all would like to see it printed in THE MATHEMATICS TEACHER, since we believe other teachers might well attempt this type of educational liaison between mathematics and the language arts. The luckier teachers may even have such pupils as Linda, whose poetry on "That Meaningless Game" we send you.

Sincerely,
LOWELL VAN TASSEL
Math Department
Herbert Hoover High School

THAT MEANINGLESS GAME

Mathematics is a meaningless game,
To some it brings a lot of shame,
I am one of whom it's true,
For I can't tell a one from a two.
Why should I waste my time
On silly things that don't even rhyme,
When I could be doing my work,
But no, I have to be a jerk.

Sometimes the game is lots of fun,
But other times from it I'd like to run.
It confuses and tires my poor little brain
Until I think it's gonna be lame.
But one thing I wish, I wish they'd explain
to me,
How can they say that x equals three,
When three is a number and x is a letter?
At things like this I'm really much better,
But I have to act dumb
So this will rhyme some.

—Linda Chamberlin, student, Herbert Hoover High School, San Diego, California.

Mosaics by reflection[1]

J. MAURICE KINGSTON, *University of Washington, Seattle, Washington.*

Infinite plane-fillings or tessellations of polygons are presented for use in high school classes and mathematics clubs.

THE COMMON EXPERIENCE of standing between two parallel plane mirrors and observing the "unending" sequence of images of oneself in each mirror leads very naturally to the question of what happens when one is surrounded by *three* vertical plane mirrors. In this case our images appear over the whole plane or over as much of it as we are able to observe, subject to the limitations imposed by our eyes and the imperfections of the mirrors.

If, instead of a person, we use a point object and draw red lines from the point perpendicular to each mirror, we find that we have "filled" the plane with a symmetric array of red polygons whose shape depends on the angles between the mirrors and the location of the object point. Although an infinitude of distinct mosaics is possible with such a kaleidoscope, we will see that a proper choice of the two variables just mentioned will lead to regular polygons in a given plane-filling. If each vertex is surrounded alike by the same set of regular (but not necessarily identical) polygons, the plane-filling is said to be semiregular or *uniform*.

I have found that the subject of infinite plane-fillings, or tessellations, by polygons was greeted with considerable interest by students on the several occasions when I have spoken on this topic before a high school mathematics class or club. Perhaps one reason lies in the ease with which the ideas involved can be demonstrated and the elegance inherent in the resulting symmetry of the mosaics.

There are exactly eleven distinct uniform plane tessellations, including three which are *regular*, i.e., all the polygons involved are regular and identical. Of these eleven uniform tessellations, only eight are *edge-reflexible*, by which we mean that each of the eight is symmetric with respect to a line through the mid-point of any edge, perpendicular to that edge. This property of being edge-reflexible permits such a tessellation to be reproduced by a simple arrangement of mirrors, as we shall see shortly.

When we have a single mirror, an object has but one image and their join is a straight line segment (Figure 1).

Figure 1

With two mirrors the situation is immediately more complex. In the first place, the reflection of each mirror in the other mirror produces a virtual mirror which acts like a mirror. This situation is then repeated over and over again. It is not difficult to verify that the two original mirrors plus this set of virtual mirrors will eventually repeat themselves whenever, and only whenever, the angle between the

[1] The material of this paper was the subject of a half-hour television program by the author on November 4, 1955, on Station KCTS, arranged in co-operation with the Bellevue, Washington, Schools.

mirrors is a rational fraction of 180°, or equivalently of π radians, e.g., $d\pi/n$ radians. Hence in such cases some image of the set will eventually overlap and coincide with the object, which situation we describe by saying the set of images is *discrete*.

It may be shown that the same pattern results for angle π/n as for $d\pi/n$, provided that d and n are relatively prime,[2] positive integers. Hence it will only be necessary to deal with angle π/n. When the object lies on one of the two mirrors (not, of course, on their intersection) the joins of object to image successively form a regular n-gon often denoted by $\{n\}$, while if the object lies on the bisector of the angle between the mirrors, the figure formed is a regular $2n$-gon or $\{2n\}$. Figures 2 and 3 illustrate these two cases for $n=3$.

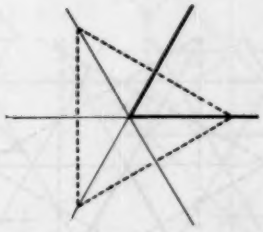


Figure 2

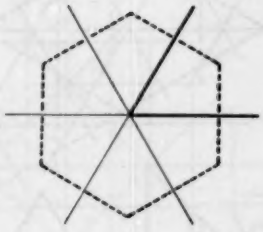


Figure 3

However, this is merely preparation for the case which is our main interest, viz., three plane mirrors, each perpendicular to a given plane (Figure 4). If the three

[2] Two integers are relatively prime if their only common factor is 1.

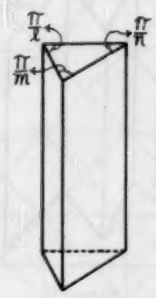


Figure 4

angles are represented by π/l, π/m, and π/n radians, then

$$\frac{\pi}{l}+\frac{\pi}{m}+\frac{\pi}{n}=\pi,$$

and there are only *three* positive, integral solutions of the equation

$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=1,$$

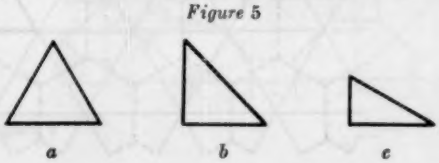
viz.,

(a) 3, 3, 3
(b) 2, 4, 4
(c) 2, 3, 6

(See Figure 5.) As shown, a convenient symbol to use to represent each of these three solutions is the actual triangular cross-section of the three mirrors. In what follows the object will then be indicated by a dot placed at the proper location; i.e., (i) at a vertex, (ii) at the intersection of an angle bisector with the opposite side, and (iii) at the intersection of the three angle bisectors.

To reproduce the three cases a, b, and c it is not necessary to have nine mirrors. It will suffice to cut four rectangular pieces of plane mirror of a convenient height, say 12 inches, and with base lengths of 3, 6, 6 and $6\sqrt{2}$ (=8.5 approximately)

Figure 5



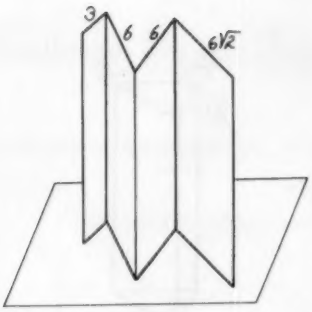


Figure 6

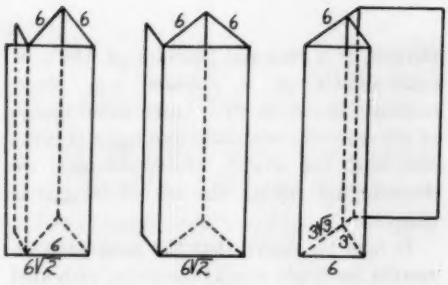


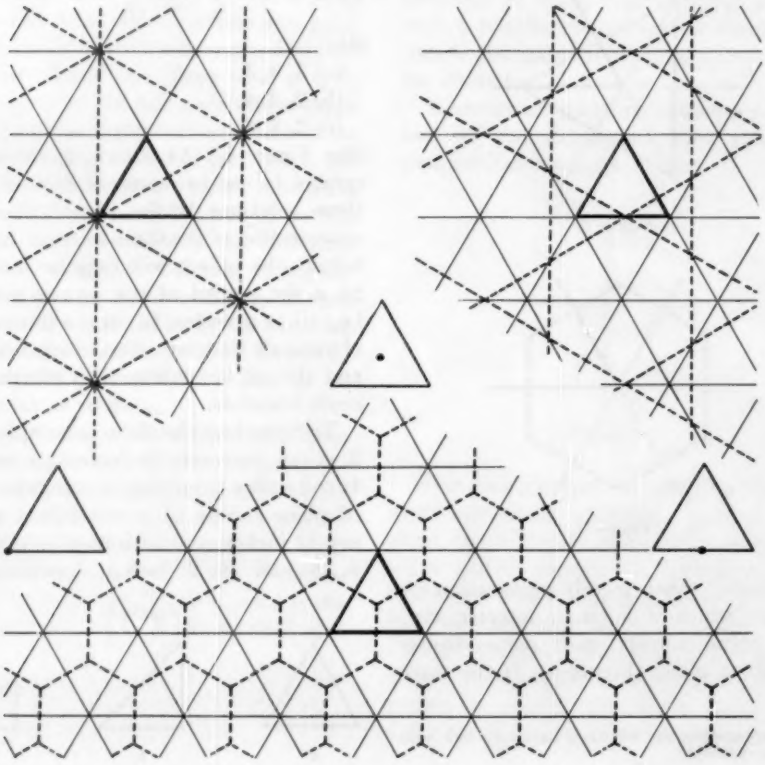
Figure 7

inches, respectively. These four mirrors are joined by strips of adhesive tape (which will act as three hinges) along their 12-inch lengths in the above order, as shown in Figure 6. Then the three cases may be set up as shown in Figure 7.

It is advisable to apply narrow strips of adhesive tape to all exposed edges of the mirrors to avoid inevitable cuts from the sharp edges and to provide a cushioning effect to prevent breakage.

Since our purpose is to produce only *regular* polygons, case (a) allows for only *three* different positions of the object point, on account of the symmetry of the solution. The object point at a vertex gives

Figure 8



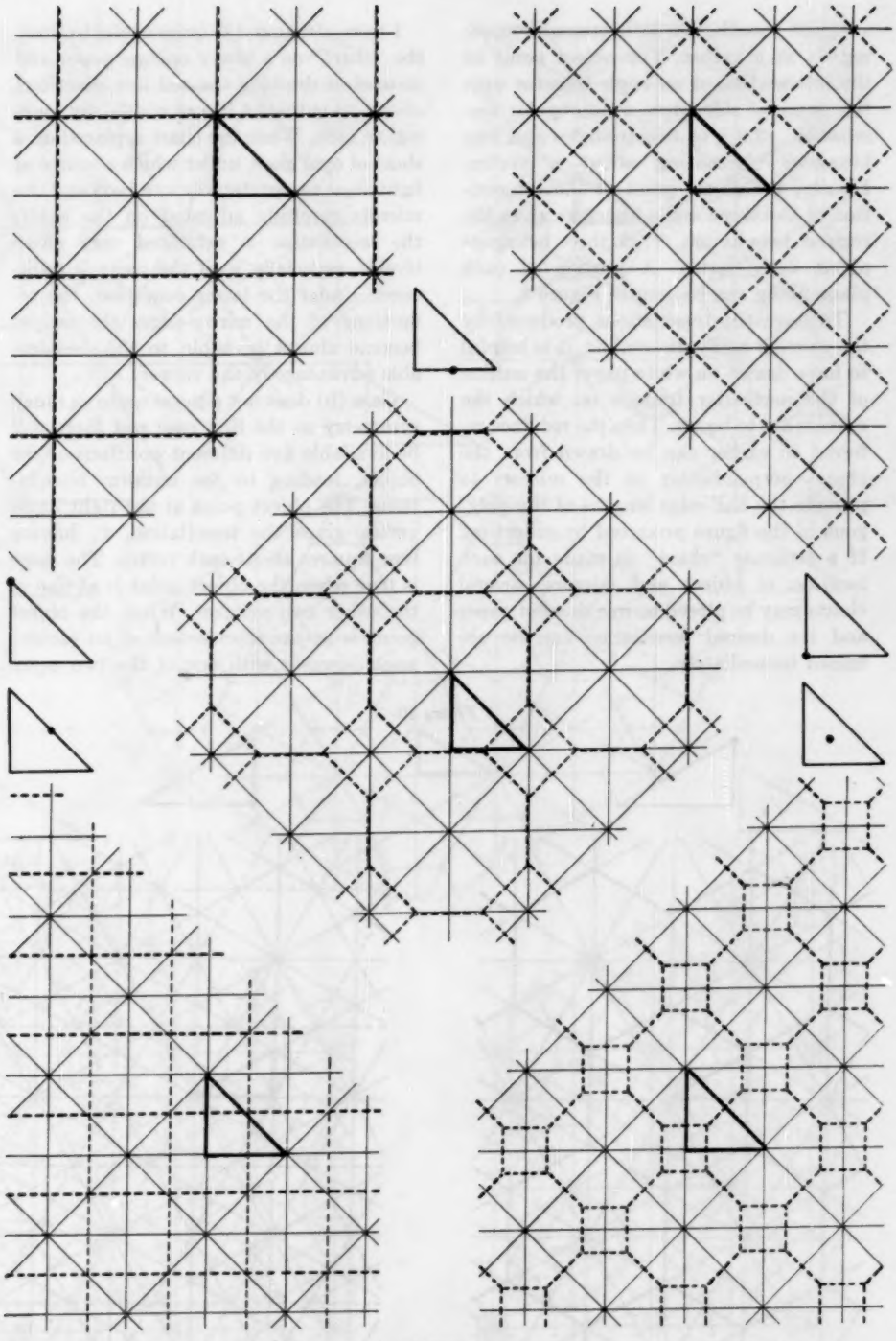


Figure 9

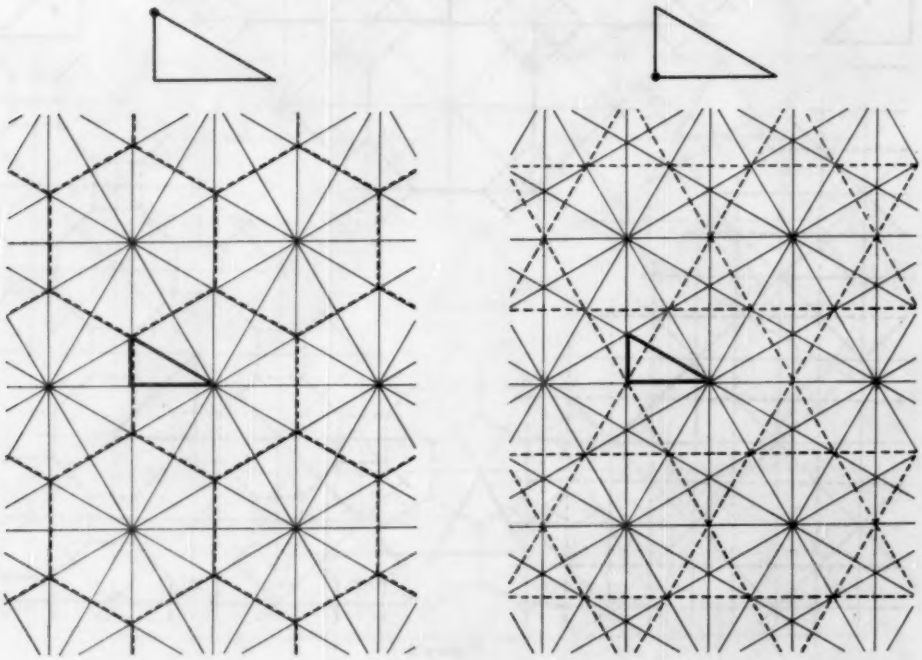
a regular tessellation, 3^6, of triangles meeting six at a vertex. The object point at the intersection of an angle bisector with the opposite side gives a nonregular tessellation, $(3.6)^2$, of two triangles and two hexagons alternating about a vertex. Finally, the object point at the intersection of the three angle bisectors gives the regular tessellation, 6^3, of three hexagons about each vertex. A portion of each plane-filling can be seen in Figure 8.

To have the tessellations produced by the mirrors easily discernible, it is helpful to have drawn on white paper the outline of the particular triangle on which the mirrors are to be set. Then the *red* lines referred to earlier can be drawn from the object perpendicular to the mirrors to provide the half-edge lengths of the polygons in the figure produced by reflection. If a separate "chart" is made for each location of object and mirrors, several charts may be placed on one sheet of paper and the desired tessellation can be obtained immediately.

I have also found it quite useful to draw the "chart" on a heavy opaque paper and instead of drawing the red line described above, to cut out a line of width, say, one-eighth inch. When the chart is placed on a sheet of opal glass, under which a source of light is set up (preferably in a box) and the mirrors carefully adjusted on the chart, the tessellation is exhibited very effectively—especially so if the room is darkened. Under the latter condition, the reflections of the mirror-edges themselves become almost invisible, to the considerable advantage of the viewer.

Case (b) does not possess quite as much symmetry as the first case and there will be available five different positions of the object, leading to *five* uniform tessellations. The object point at the right angle vertex gives the tessellation, 4^4, having four squares about each vertex. The same is true when the object point is at one of the other two vertices. When the object point is at the intersection of an (acute) angle bisector with one of the two equal

Figure 10



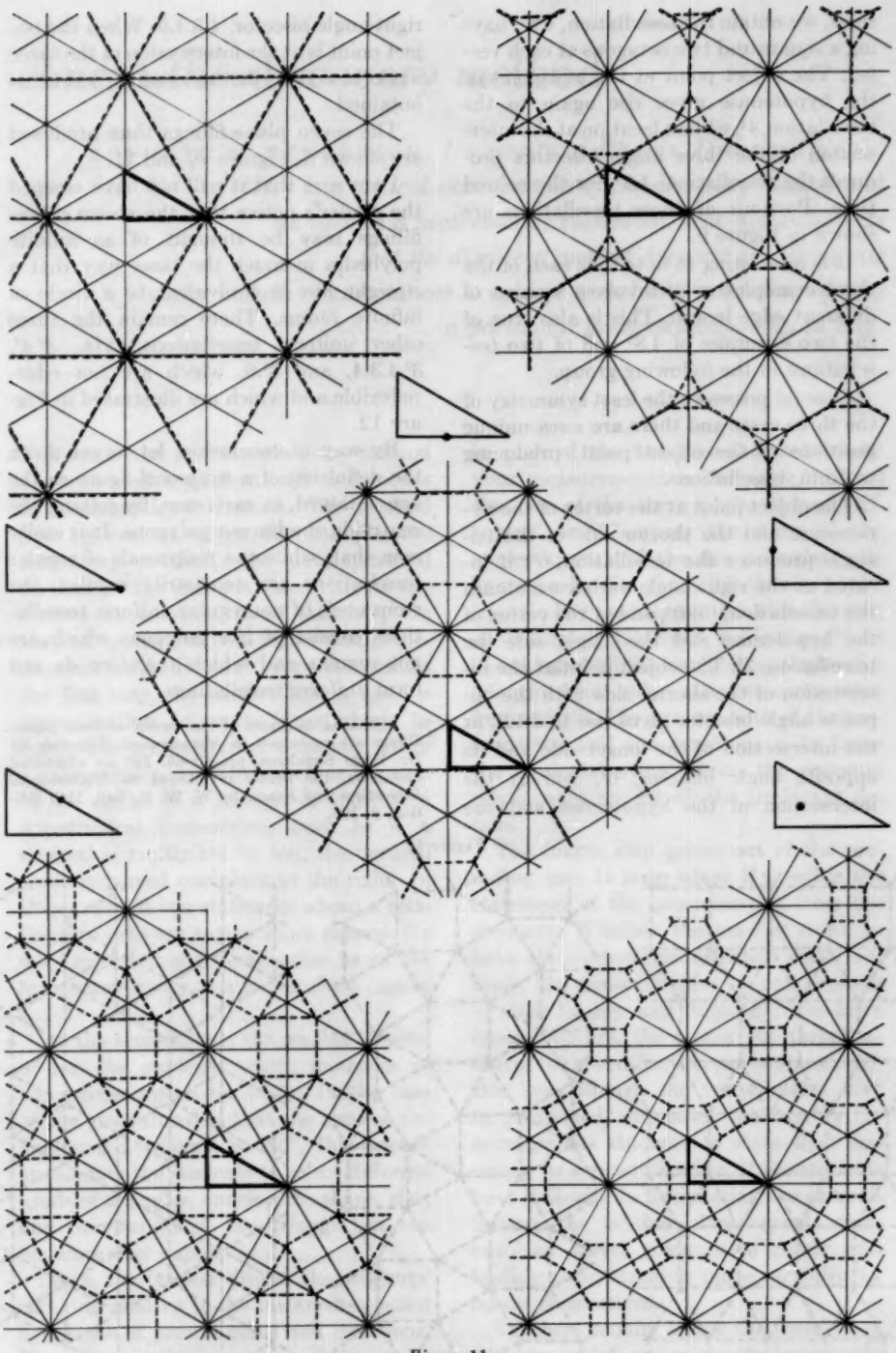


Figure 11

sides, we obtain the tessellation, 4.8^2, having a square and two octagons at each vertex. The object point at the mid-point of the hypotenuse gives rise again to the tessellation, 4^4, while a location at the intersection of the three angle bisectors produces the tessellation, 4.8^2, for the second time. Portions of these tessellations are shown in Figure 9.

It is interesting to note that each of the three examples of 4^4 involves squares of different edge length. This is also true of the two examples of 4.8^2 and of two tessellations in the following group.

Case (c) possesses the least symmetry of the three cases and there are *seven* unique positions of the object point, producing uniform tessellations.

The object point at the vertex of the hypotenuse and the shorter side of the triangle produces the tessellation, 6^3; if located at the right angle vertex we obtain the tessellation, $(3.6)^2$, and at the vertex of the hypotenuse and the longer side the tessellation, 3^6. The object point at the intersection of the shorter side with the opposite angle bisector gives rise to 3.12^2, at the intersection of the longer side and its opposite angle bisector, 6^3, and at the intersection of the hypotenuse and the right angle bisector, 4.3.4.6. When the object point is at the intersection of the three angle bisectors, the tessellation, 4.6.12, is obtained.

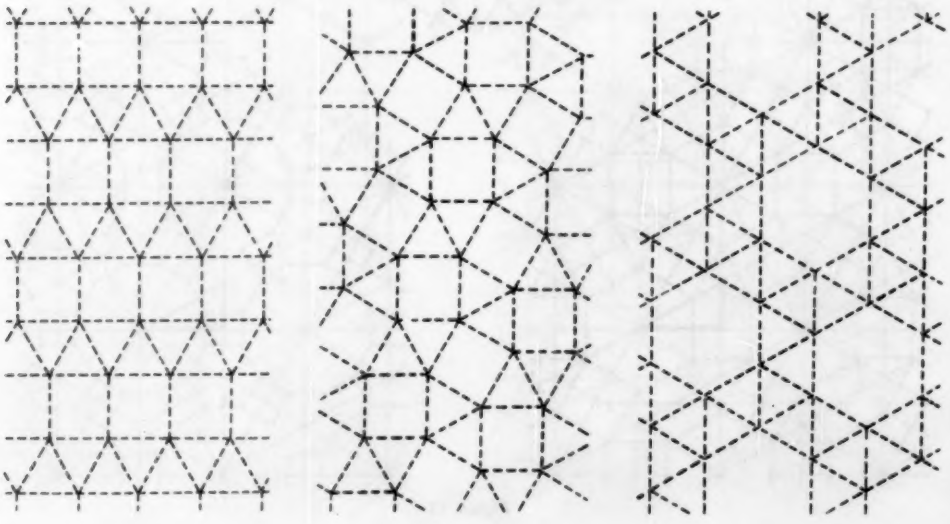
The seven plane-fillings thus produced are shown in Figures 10 and 11.

I am sure that it will not have escaped the reader's notice that the above plane-fillings may be thought of as infinite polyhedra in much the same way that a straight line is equivalent to a circle of infinite radius. There remain the three other uniform tessellations,[3] viz., $3^3.4^2$, $3^2.4.3.4$, and $3^4.6$, which are not edge-reflexible and which are illustrated in Figure 12.

By way of conclusion, let us set down the definition of a *reciprocal* figure as the one obtained, in each case, by joining the centroids of adjacent polygons. It is easily seen that, while the reciprocals of regular tessellations are necessarily regular, the reciprocals of nonregular uniform tessellations consist of *like* polygons which are not regular and which therefore do not form uniform tessellations.

[3] Striking diagrams of the eleven uniform plane-fillings are contained in *Mathematical Snapshots* by Dr. Hugo Steinhaus, pp. 60–65. For an additional discussion, the reader is referred to *Mathematical Recreations and Essays* by W. W. R. Ball, 11th Edition, p. 106.

Figure 12



Anent the discovery method

KENNETH B. HENDERSON, *University of Illinois, Urbana, Illinois.*

Here is a critical examination of the discovery method and its uses in a mathematics classroom. The author asserts that the discovery method does not enable pupils under certain circumstances to achieve a real understanding of why a particular generalization may be true.

FOR SOME TIME NOW the discovery method has been receiving attention from theorists in mathematics education. The method goes by various names, for example, the laboratory method, the developmental method or the inductive method. As described by some of these theorists, the method appears to be a formalized procedure not unlike Herbart's formal steps. As the first step, the teacher has in mind a generalization he wants his students to learn. The generalization is usually either a categorical proposition, such as the sum of the angles of a triangle is 180°, or a hypothetical proposition, such as if a decimal is multiplied by ten, the decimal point is moved one place to the right. In either case, it is a statement about a relationship between two or more classes. We may regard this generalization as of the form where every case of p is also a case of q.

As the second step, the teacher selects, or has the students select, instances of the generalization for study. In the case of the generalization that the sum of the angles of a triangle is 180°, the teacher encourages the students to select different kinds of triangles, varying the shape, size, and orientation of the triangle on the blackboard or paper.

Next, the teacher directs the students' thinking relative to the instances selected by means of prescriptions and questions. He may say, "Measure each of the angles in each of the five triangles and record your measurements in the table," "Find the sum of the three angles in each triangle," or "Compare the sums of the angles in each of the five triangles." Or he may ask, "Do you think the size of the triangle has anything to do with the sum of its angles? How can we find out?" or "Looking at all the sums of the angles of a triangle, what value seems to occur most frequently?" The purpose both of the prescriptions and the questions is to focus the students' attention on the common relationship amidst all the irrelevant details.

The fourth step grows out of the preceding one. It is to educe if possible the statement of the generalization from the students. Whether the teacher seeks to have the students immediately state precisely the generalization or not depends on their facility with language. The more facile they are, the sooner the verbalization of the generalization can be attempted. But even though the verbalization may be postponed, ultimately the teacher encourages the students to state their discovery as a generalization. If the students have difficulty in formulating the generalization, the teacher often repeats steps two and three, going more slowly and leading their thinking more carefully by means of questions.

Teachers usually follow the fourth step with one which makes the students apply

the generalization they have just acquired in new situations or problems. This has the effect of extending the generalization.

Let us now look at the logic of the discovery method. The logic is that of the logic of a probable inference, for when a student predicates to the entire class that relationship he has identified as true of the sample with which he has been working, he makes a probable inference.

As was pointed out above, the generalization may be regarded as being of this form: Every case of p is also a case of q regardless of other contingencies. The following may be regarded as a paradigm of the argument for the material truth of a generalization expressed in this form.

Let us represent "being a triangle" by "p," "having 180° as the sum of the interior angles of the triangle" as "q," "being an acute triangle" as "r," "being a scalene triangle" as "s," "being so oriented on paper that the longest side is toward the bottom of the paper" as "t," and so on—identifying all the factors we think might be relevant.

1. Each instance (measurement or comparison) which confirms the generalization is essentially a report that in addition to p and q, other factors such as r, s, and t, occur or fail to occur.

2. Both p and q, but no other factor, occur in all instances.

3. No instances are found in which p occurs but no corresponding q, that is, no contrary evidence turns up.

4. Therefore ("therefore" is used in the sense of "it probably follows" and not in the sense of "it necessarily follows") every case of p is also a case of q regardless of other contingencies.

This is the familiar method of agreement. (The method of difference can be used in employing the discovery method or the joint method of agreement and difference, depending on the nature of the generalization to be "discovered.") The persuasiveness of the paradigm may be attributed to two factors: (1) its application affords an accumulation of confirmatory instances, and (2) factors r, s, t, et al. are shown not to be relevant.

In the empirical sciences, correct use of the method of agreement depends on the investigator's ability to identify all possible relevant factors. In a body of analytic truth like mathematics, this is no problem for the teacher. He knows for certain the factors and only those factors which are relevant. The students who are in the role of investigators do not have this knowledge, hence the teacher may help them by such leading questions, as "Do you think the shape or size of the triangle has anything to do with the sum of the angles? How can we find out?"

The generalization limited to the sample of instances is an enumerative generalization. The criterion for its verification is clear. It is true for the sample if and only if it is true for every member of the sample. But if the generalization is limited to the sample, it has little power. It needs to be extended to the class from which the sample was drawn. But as Hempel[1] has pointed out, there are no criteria to establish the confirmation of an empirical generalization about a population (class) which is larger than the class for which the generalization has been verified. To apply this to the example we have been using, when the teacher limits his teaching to the discovery method, there is no way of verifying the generalization which the student is supposed to reach. The teacher does not even know how many triangles he should have the students measure to confirm the generalization. Acceptance of the generalization is based on a psychological rather than a logical criterion.

Let us now look at the psychology of the discovery method from the point of view of the student. If the student is told by the teacher what the objective of the study is, namely, to develop a generalization from the comparisons or measure-

[1] Carl G. Hempel, "Studies in the Logic of Confirmation," *Mind*, LIV (1954), 1–26; 97–121.

ments he makes, his thinking is structured to some extent. In analyzing his thinking, three concepts are useful. The first of these is that of logical form. One kind of logical form is a relationship, expressible in language, between two or more classes of elements. Such a logical form is an intellectual construct or abstraction which characterizes the instances of it.

The second concept is that of content. The content is the medium in which a logical form is expressed. To turn to the example we have been using, the content is the class of triangles of different sizes, shapes, and orientations which the student studies. But all have the common logical form or property that the sum of the interior angles is 180°. The same form is expressed in different content. As we have seen, it is this logical form that the teacher wants to teach. He varies the content in which this form appears so the students can see the common logical form in differing content.

The third concept is that of analogy. Analogy is the recognition of a common form in different content. An analogy is an inference. When the student sees the correct analogy, he makes the discovery, or, as we sometimes say, has the insight. His ability to see the analogy is determined to some extent by his native ability and to some extent by whatever previous training he has had designed to help him understand the process he is going through. He proceeds by formulating verbally or nonverbally hypotheses (inferences) which he tests. He tests these by means of various data which are either relevant to the logic of the process or are extraneous to the logic. Examples of the former are the comparisons or measurements he has made or makes anew. Examples of the latter are the cues he gets from the teacher's facial expressions, tone of voice, or verbal responses. As a result of the testing, he either accepts or rejects the hypothesis. If it is the latter, he searches for a new hypothesis and the whole procedure is re-enacted. Once he accepts the hypothesis, he has made the discovery. If he has studied several members of the class and if he does not understand the logic of the method of induction, he readily makes the some-to-all extension and predicates to the entire class that relationship which he has found true of the sample with which he worked.

A strong case can be made for the discovery method at least pragmatically. Teachers who have tried it testify to the great interest the students maintain. They also feel that the quality of the students' thinking is higher than in the telling method. A third advantage is that in using the discovery method, it is hard to go too fast for students. The teacher usually is aware whether the students have made the discovery. A smile or an "I see" are often cues which the teacher can use to detect the insight. If the insight has not been manifested, the teacher retraces his steps, reducing the diversity of the content and guiding the students more carefully by leading questions. These cues are not so readily available in the telling method, hence it is easy to go too fast for the students. Finally, it is felt by some theorists that when the student makes the discovery for himself, the generalization is more meaningful and better understood.

So often the discovery method is made to glitter by comparing it with the drab kind of teaching that sometimes is done when the telling method is used. I have never felt that such a loaded comparison is quite fair, for so often a description of an ineffective use of the telling method is given. The teacher using this method is portrayed as presenting a straight lecture without setting the stage for learning by showing how what the students are to learn is related to what they already know, without checking to see if the students understand the terms that are used in the presentation, and without giving them a chance to apply the generalization they have learned.

Using an unfair comparison, one could just as easily make the user of the discovery method look bad. He could be por-

trayed as not leading the students' thinking effectively, as forcing them to generalize from two or three instances, as allowing the bright students to state the generalization as soon as they conceive it, thereby immediately destroying the motivation of the other students, or as furnishing extraneous cues by facial expression or tone of voice which enable the students to test their hypotheses. Let us realize that the discovery method can be abused just as the telling or deductive method can be abused. And in my opinion it would be abused if students subtly were taught to generalize from a few instances or if they were just as subtly taught to pay attention to the teacher's reactions rather than to the data which become relevant once the hypothesis is formulated. After observing the teaching of some teachers who say they use this method, I fear that some of their students may be acquiring these undesirable learnings.

One further point about the discovery method is bothersome. Suppose suddenly all mathematics teachers were to shift from the telling or deductive method to the discovery method. Would not the students come to think that mathematics is like science in that we arrive at generalizations in both subjects by using the same method? Might they not come to think that mathematics, like science, is a body of synthetic or contingent truth? Of course, mathematics is not this kind of truth. It is not so bad if students in the elementary school and junior high school acquire some generalizations by the discovery method and never verify them deductively. But sometime in high school they should learn the distinctive nature of mathematics as a system of postulational thinking. They should learn the different logic of the methods of induction and deduction and when each method is appropriate. Conventionally this is done, if it is done at all, in demonstrative geometry. But it does not necessarily have to be learned in this subject alone. It can be learned initially in ninth-grade algebra and re-enforced in succeeding courses. It would seem that the farther the student goes in high school mathematics, the more the deductive method should be used. The discovery method can be used to identify hypotheses, but these in most cases should then be proved or disproved deductively.

It would seem that the relative advantages of different methods of teaching should be appraised largely on the depth of understanding their use affords. It helps to identify six levels of understanding. Let us for simplicity's sake assume that we are interested in teaching a generalization. The first level is represented by a student who can give a verbatim statement of the generalization which he has memorized from the textbook or from the statement the teacher has written on the blackboard. He does not know what all the words in the generalization mean, and he cannot give an instance of it. This certainly is a low level of understanding. Some people may even not want to call this behavior understanding at all, since there seems to be no meaning for the student. But such a student probably could recognize some permutations of the words in the generalization as nonsense. For example, such a student who could repeat "the sum of the angles of a triangle is 180°" without knowing what an angle or a degree is, might say the progression of words, "the is of the 180° triangle angles sum" is nonsense. Were he to say this, I would be willing to say the generalization had some sort of syntactic meaning for him, meager though it is.

The second level is represented by a student who can give a correct statement of the generalization in his own words. He cannot give an instance of the generalization or apply it, but his ability to paraphrase it correctly seems to indicate that it has a syntactic meaning for him which goes beyond the meager syntactic meaning of the first level.

The third level is represented by a student who, in addition to being able to perform at the second level, can give instances

of the generalization, instances which are not those that the teacher or the textbook has given. We might say that the generalization has a semantic meaning for this student.

The fourth level is represented by a student who goes beyond the third level by being able to apply the generalization in problems for which he knows the generalization is relevant. Perhaps the teacher or the textbook has stated, "Find the third angle in a triangle, two of whose angles are 90 degrees and 50 degrees" directly after teaching the generalization about the sum of the angles of a triangle. In this case the student would have to be quite incapable not to guess that the generalization he had just learned had something to do with the problem.

The fifth level is represented by a student who, besides being able to operate at the fourth level, can apply the generalization in problems where there is no second party to give him a cue that the generalization is relevant. For example, such a student should be able to solve the problem, "One angle of a triangle is twice the second angle and three times the third angle. Find the size of each angle of the triangle."

The sixth and final level is manifested by a student who not only can operate at the fifth level, but also can explain or prove the generalization in the sense of showing that it is necessarily implied by certain other propositions.

Although I have no conclusive evidence, I would hazard a guess that the discovery method enables a student to move readily to what I have arbitararily designated as the fourth level of understanding. Perhaps it will do this even better than the telling method, but I am not sure. But if the teacher limits his teaching to the discovery method, I doubt that his students ever will reach the sixth level, the stage of understanding at which they can give reasons why the generalization is true for all the instances for which they apply it. When I say this, I am not implying that I believe the sixth level should be aimed at for every generalization. I doubt that it should. I am simply indicating what I believe is an inherent weakness of the discovery method as far as mathematics teaching is concerned.

I would like to close with a reaffirmation of my belief in the efficacy of the discovery method. I have voiced some reservations about the method because of the great attention the method receives in professional journals, if not in the classroom. I fear that teachers who are looking for a panacea to solve their problems of methodology might uncritically climb on the bandwagon. What the classroom teacher needs is an understanding of and an ability to use several different methods, for to date there is no royal road to teaching and learning.

Have you read?

GALBREATH, J. W., "The Critical Shortage of Science and Mathematics in Our Schools Today," *School Science and Mathematics*, November, 1956, pp. 595–601.

This topic is "old hat," but you will be interested in the author's reasons for the situation, what could be done, and what is being done. For example, has science made the world in which we live so easy that talent is not challenged for better things? Maybe we have failed as salesmen; perhaps students are different; or do we move so rapidly that education can't keep up? These and other reasons are discussed.

What can be done? You will be interested to know the author feels some things which are distasteful are often challenging; that intellectual accomplishment has certain satisfactions; that "whys" are often the key to motivation, and that teacher background and knowledge need much consideration. He offers some suggestions as to how all this might be accomplished. You will also want to read about some things in actual practice which you may even want to copy.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Mathematical games build skills

by Donovan A. Johnson, University High School, Minneapolis, Minnesota, and Clarence Olander, St. Louis Park High School, St. Louis, Minnesota

In the same way that athletic contests inspire students to spend tremendous time and energy in practice, classroom games may be a means for stimulating learning activities. Why not capitalize on your students' interest in the activity and competition of games to promote learning and practice activities that are exciting? When mathematics practice is an adventure most of our difficulties in building mathematical skills vanish.

Games are usually a recreational activity. They are often a means for releasing the strains and tensions of our daily routines. At the same time, games may be a way to develop restraint, to be observant, to follow rules, to practice diligently. Games also give students opportunities to work co-operatively with a group, to contribute as a leader and follower, to accept responsibility for individual as well as group actions. *Of course learning mathematics is not all a game.* Other types of individual and independent activities are essential.

What is the role of mathematical games?

As a mathematics teacher, you can use games to achieve purposes such as:

1. *Games can be used to make practice periods pleasant and successful.* Probably the most difficult aspect of a practice exercise is the prevention of boredom and monotony. One way to add variety to practice is through the use of games. In practice involving common and decimal fraction equivalents, why not play "Old Number," a game like "Old Maid"? In this game the "pairs" would consist of decimal and fractional equivalents. The Old Number is ten!

2. *Games can be used to teach vocabulary.* Several games can be based on mathematical words. For example, a Rummy game may use cards which list the technical vocabulary you wish your students to master. Another possibility is to write mathematical words on the board with the letters in mixed order. The students unscramble the words and match them with appropriate pictures or problems.

3. *Games can be used to teach mathematical ideas.* A game such as "Battleship" is a very realistic way to learn the meaning of co-ordinates in graphing. In this game, shots are made at points identified by co-ordinates.

4. *Effective study habits may be motivated through the use of games.* When students know that they will be participating in a group game in which they will be expected to contribute, they will be motivated to become proficient.

5. *Games are useful in providing for individual differences.* Your pupils can be en-

couraged to devise and make games of their own choosing. Try challenging pupils to make up card games, spelldowns, bingo games, word association games, "Dr. I.Q. Quizzes" or "What's My Line." Games can be adjusted to the interests and abilities of small groups of students and consequently be used to good advantage with the slow learner, average student, or gifted student.

6. *The attitude of children towards mathematics can be favorably changed through the use of games.* Too many of us expect our pupils to have the same attitude toward mathematics that we have! Games tend to throw more life and informality into what might otherwise become a drab and formal situation.

7. *Games can be used to improve reading in mathematics.* Many games contain problems or procedures that require reading similar to that of the mathematics textbook. Some game situations require rapid reading with comprehension similar to that of tachistoscope presentations.

8. *Games can provide a means of summarizing or reviewing a unit.* For example, a game like bingo can be made up with cards having algebraic expressions. The call cards each have two factors, while the playing cards have the products of these factors.

9. *Games can be related to seasonal events, and hence add to the enjoyment of the classwork.* During football season, play mathematical football in the classroom! A list of problems on separate slips should be prepared and drawn by the team that "has the ball." The yardage advanced should be in proportion to the difficulty of the problem. Each pupil takes his turn at "carrying the ball." Fumbles, field goals, extra points, may all be involved in the rules.

10. *Games may be appropriate "homework."* A game can be an ideal situation for parents to encourage out-of-class learning. A game also is an excellent way for student and parents to work together at learning mathematics. Hence an excellent public relations situation develops.

How are mathematical games properly used in the classroom?

The success of a classroom game, like any instructional material or technique, is highly dependent on how it is used. If a game is to play the roles described above, it should be used at the right time, for the right purpose, in the right way.

1. *The game to be used should be selected according to the needs of the class.*

The basic criterion requires that the game make a unique contribution to learning that cannot be attained as well or better by any other material or technique. The material involved should be closely related to that of the regular classwork. Whatever game is selected, it should involve important mathematical skills and concepts. Major emphasis should be on the learning of mathematical concepts or skills, rather than on the pleasure of playing the games.

During the game situation all students must participate. Even though only one person is working on a certain problem, every team member must be responsible for its solution too. Games must also avoid extreme embarrassment by the person who cannot solve a problem. Whenever possible, each student should compete with his peers, and should be working on material suited to his ability.

2. *The game should be used at the proper time.* If games are to make the maximum contribution to the learning of mathematics, they should be used during the regular class period when the ideas or skills are being taught. Many teachers, however, prefer to use games after completing a topic, on the day before a vacation, during the students' free time at noon, or in study periods or homerooms. Others use them during days of heavy absence due to athletic games, storms, concerts, or excursions. Some teachers use them as rewards for work well done, while others use them for remedial work. Usually games should be relatively short so that pupils do not lose interest.

3. *The game must be carefully planned*

and organized so that the informality and excitement of the setting does not defeat its purpose. Students may often plan games as well as referee the competition. Before beginning the game, the participants should be briefed on the purpose of the game, the rules of the game, and the way to participate. Often the students can establish ground rules, so that everyone (including the teacher) may enjoy the activities. "Coaching" or "kibitzing" should not be allowed. The loss of points for the breaking of rules is usually sufficient to maintain appropriate behavior. Avoid the choosing of team members by student captains, since low ability pupils should not be embarrassed by being chosen last.

4. *The participants in the game must accept the responsibility of learning something from the game.* Follow-up activities such as discussions, readings, or tests will emphasize this responsibility. As the teacher, you need to evaluate the results by asking yourself how successful the game was in promoting desired learning.

What games can be used to learn mathematics?

Many mathematics games can now be purchased ready for use in the classroom. However, many games can be prepared by students or teachers. Making games based on current class work may in itself be an excellent learning experience.

Games appropriate for learning mathematics are limited only by the ingenuity of students and teacher. Most lessons involving practice can be converted into a learning game by choosing teams, participating as individuals or teams, and keeping score. Most of the common parlor game or athletic game rules can be adapted for use in a mathematics class at any grade level. Quiz games, card games, and word association games have been successfully used by many teachers.

Printed cards with numbers on them, such as Rook or Flinch cards, are often useful. Dice or spinning discs can also be used to supply numbers. For example, a pair of dice can be used to determine the location of an ordered pair of numbers on a co-ordinate plane. Another interesting variation for dice is to make and use regular polyhedrons such as octahedrons, dodecahedrons, or icosahedrons. Numbers, algebraic expressions, or problems can be written on the faces of these polyhedrons depending on the game involved. Ordinary three by five inch cards are inexpensive and easy to label. Crayons, china marking pencils, or brush pens are suitable for marking the cards. Blank playing cards obtained from a magic shop will add attractiveness to your game.

Games are appropriate in your mathematics classroom if you consider a systematic program of practice essential. It is true that drill activities were emphasized too much in the mathematics classroom of a generation ago. Practice and drill were carried to extremes because it was considered sufficient to gain complete learning. When it was found that this skill learning resulted in poor retention, little understanding, and almost no application in daily problems, practice was recklessly abandoned. It was decided that the key to learning mathematics was through meaningful experiences and practical applications. As is usually the case, the best approach to learning mathematics seems to be a compromise which uses meaning, practice, and application. The meaning of numbers, the understanding of a process, the mathematical structure involved precede the practice. Practice then is the part of the learning process which builds accuracy, efficiency, and retention. If your provide the proper amount of practice at the appropriate time, it is likely that many of your students will have the mathematical competence that business, science, industry and colleges are demanding. So with the help of students and the use of some imagination you are all set to start an adventure. Don't be surprised if your students ask for mathematics games to play at noon hours or at home!

• MEMORABILIA MATHEMATICA

Edited by William L. Schaaf, Brooklyn College, Brooklyn, New York

Mathematics and morals . . .

Historically, religious education and training have never been regarded as proper functions of state-supported schools in the United States. At the same time, however, character education has always been an implicit major objective of our schools. The public rightly expects the schools to be concerned with the development of moral values and spiritual ideals. Character-building is a legitimate outcome of the curriculum, in general and in particular.

Obviously, some curriculum areas afford greater opportunities for the cultivation of moral and spiritual values; most of the major subjects can contribute something of significance—social studies, language arts, science and mathematics, fine arts, industrial arts, home economics and family living, vocational education, health and physical education—not to mention many related activities not directly connected with any particular curriculum. These contributions are ably set forth in an engaging pamphlet[1] recently issued by the Board of Education of the City of New York. Because our readers will want to know, we quote below what is said (pp. 11–12) about the contribution of science and mathematics to character education:

> The study of science and mathematics involves a process of discerning order, system, balance and law in the wonderful phenomena of the biological, chemical and physical world. Scientists and mathematicians conceive of the universe as a logical, orderly, predictable place. The vastness and the splendor of the heavens, the order and precision of the sun, planets, stars and comets, the marvels of the human body and mind, the beauty of nature, the mystery of photosynthesis, the mathematical structure of the universe, the concept of infinity cannot do other than lead to humbleness before God's handiwork.
>
> Science and mathematics cannot create life but they can be instruments through which there is gathered, for the advancement and comfort of the human race, the knowledge and techniques for using and controlling the elements and forces of a bounteous universe. Properly taught, mathematics and science can contribute to the realization of the necessity for the close co-operation of social and scientific agencies as institutions for serving mankind.
>
> The arithmetic of the home, of banking, and of tax payments offers countless opportunities for developing moral responsibility. Properly taught, arithmetic can point out the dishonesty of living beyond one's means, the responsibility for sharing in family income and expenditures, and the value of thrift.
>
> Both science and mathematics, by their very methods, lay the groundwork for developing a devotion to truth. They develop ability to examine facts and evaluate evidence, to maintain open-mindedness, intellectual honesty, and suspended judgment, and to reach unbiased conclusions. On the other hand, pupils who become familiar with the methods and wonders of science realize that in many areas scientific experimentation is impossible, and that the application of the methods and procedures of science has definite limitations.

[1] *The Development of Moral and Spiritual Ideals in the Public Schools.* Board of Education, New York, N. Y., 1956. 16 pages.

Note on the dedication of logarithms . . .

It has long been our conviction that the history of mathematics affords a rich source of material, which, suitably selected, might well dramatize the cultural significance of mathematics. Not the least effective of such materials would include abstracts of original sources, especially works of the seventeenth and eighteenth century. Such abstracts can be made widely accessible by means of facsimiles, as has been suggested by Professor Henry Syer of Boston University. Nor need they be limited to the works of out-

standing creative mathematicians; they might well include some of the lesser writers and commentators.

These earlier works may tell their story in several ways. For one, the very title pages themselves often reveal, in their quaint style, the hallmarks of bygone days. Again, the various *Dedications*, *Prefaces*, *Introductions*, and the occasional "*To the Gentle Reader*," are laden with human interest, and the thoughtful student, reading between the lines, may glean many valuable historical sidelights from such pages.

The story of how logarithms were invented is too well known to be retold here. Yet it is probably no exaggeration to suggest that the invention was received in the same breath-taking and awesome climate as that in which, during the past few years, the world received another unbelievable improvement in the art of computation—the advent of electronic computers: each presumably an equally fabulous achievement to its contemporaries.

We are impressed first of all, by the dedicatory epistle with which the celebrated Scottish Baron addressed his invention of logarithms to the world, as taken from his *Memoirs*[2]

> "To the Most Noble and Hopeful Prince CHARLES, only son of the High and Mighty James, King of Great Britain, France, and Ireland; Prince of Wales, Duke of York and Rothesay; Great Steward of Scotland, and Lord of the Islands.
>
> "MOST NOBLE PRINCE. Seeing there is neither study nor any kind of learning that doth more actuate and stir up generous and heroical wits to excellent and eminent affairs; and contrariewise, that doth more deject and keep down sottish and dull minds than the mathematics; it is no marvel that learned and magnanimous princes in all former ages have taken great delight in them, and that unskilful and slothful men have always pursued them with most cruel hatred, as utter enemies to their ignorance and sluggishness. Why then may not this my new invention (seeing it abhorreth blunt and base natures,) seek and fly unto your Highness' most noble disposition and patronage? and especially seeing this new course of LOGARITHMS doth clean take away all the difficulty that heretofore hath been in mathematicall calculations, (which otherwise might have been distastful to your worthy towardness,) and is so fitted to help the weakness of memory, that by means thereof it is easy to resolve more mathematicall questions in one hour's space, than otherwise by that wonted and commonly received manner of sines, tangents, and secants, can be done even in a whole day. And, therefore, this invention (I hope) will be so much the more acceptable to your Highness, as it yieldeth a more easy and speedy way of accompt. For what can be more delightful and more excellent in any kind of learning than to dispatch honourable and profound matters, exactly, readily, and without loss of either time or labour. I crave, therefore, most gracious Prince, that you would, according to your gentleness, accept of this gift, though small and far beneath the height of your deservings and worth, as a pledge and token of my humble service. Which, if I understand you do, you shall, even in this regard only, encourage me, *that am now almost spent with sickness*, shortly to attempt other matters perhaps greater than these, and more worthy so great a Prince. In the mean while, the Supreme King of Kings, and Lord of Lords, long defend and preserve to us the great lights of Great Britain, your renowned parents, and yourself, the noble branch of so noble a stem, and the hope of our future tranquility. To Him be given all honour and glory.
>
> "Your Highness' most devoted Servant,
> "JOHN NEPAIR."

It will of course be recalled that Napier died, and the completed work appeared posthumously in 1619, with notes, etc., by Henry Briggs, under the title of *The Construction of the Wonderful Canon of Logarithms*. From the translation made by William Rae Macdonald[3] we quote the prefatory letter written by Robert Napier, the most illustrious of the inventor's sons:

> TO THE READER STUDIOUS OF THE MATHEMATICS
> GREETING.
>
> Several years ago (Reader, Lover of the Mathematics) my Father, of memory always to be revered, made public the use of the Wonderful Canon of Logarithms; but, as he himself mentioned on the seventh and on the last pages of the Logarithms, he was decidedly against committing to types the theory and method of

[2] Mark Napier, *Memoirs of John Napier of Merchiston, etc., etc., with a History of the Invention of Logarithms* (Edinburgh & London: William Blackwood and Thomas Cadell, 1834), pp. 380–381.

[3] (Edinburgh and London: William Blackwood & Sons, 1888), pp. A2–A4.

its creation, until he had ascertained the opinion and criticism on the Canon of those who are versed in this kind of learning.

But, since his departure from this life, it has been made plain to me by unmistakable proofs, that the most skilled in the mathematical sciences consider this new invention of very great importance, and that nothing more agreeable to them could happen, than if the construction of this Wonderful Canon, or at least so much as might suffice to explain it, go forth into the light for the public benefit.

Therefore, although it is very manifest to me that the Author had not put the finishing touch to this little treatise, yet I have done what in me lay to satisfy their most honourable request, and to afford some assistance to those especially who are weaker in such studies and are apt to stick on the very threshold.

Nor do I doubt, but that this posthumous work would have seen the light in a much more perfect and finished state, if God had granted a longer enjoyment of life to the Author, my most dearly loved father, in whom, by the opinion of the wisest men, among other illustrious gifts this showed itself pre-eminent, that the most difficult matters were unravelled by a sure and easy method, as well as in the fewest words.

You have then (kind Reader) in this little book most amply unfolded the theory of the construction of logarithms, (here called by him artificial numbers, for he had this treatise written out beside him several years before the word *Logarithm* was invented,) in which their nature, characteristics, and various relations to their natural numbers, are clearly demonstrated.

It seemed desirable also to add to the theory an Appendix as to the construction of another and better kind of logarithms (mentioned by the Author in the preface to his *Rabdologiae*) in which the logarithm of unity is 0.

After this follows the last fruit of his labours, pointing to the ultimate perfecting of his Logarithmic Trigonometry, namely certain very remarkable propositions for the resolution of spherical triangles not quadrantal, without dividing them into quadrantal or rectangular triangles. These propositions, which are absolutely general, he had determined to reduce into order and successively to prove, had he not been snatched away from us by a too hasty death.

We have also taken care to have printed some Studies on the above-mentioned Propositions, and on the new kind of Logarithms, by that most excellent Mathematician *Henry Briggs*, public Professor at London, who for the singular friendship which subsisted between him and my father of illustrious memory, took upon himself, in the most willing spirit, the very heavy labour of computing this new Canon, the method of its creation and the explanation of its use being left to the Inventor. Now, however, as he has been called away from this life, the burden of the whole business would appear to rest on the shoulders of the most learned *Briggs*, on whom, too, would appear by some chance to have fallen the task of adorning this Sparta.

Meanwhile (Reader) enjoy the fruits of these labours such as they are, and receive them in good part according to your culture.

Farewell,
ROBERT NAPIER, Son.

It is perhaps not easy for us today to appreciate the great boon that logarithms were to navigation. A bit of evidence in this connection is the dedicatory letter of the volume of memoirs already alluded to:

TO HIS MOST EXCELLENT MAJESTY
WILLIAM THE FOURTH
KING OF GREAT BRITAIN AND IRELAND
&c &c &c

Sir,

By your Majesty's most gracious permission, I have the honour to be present to your Majesty the Domestic History of the Inventor of Logarithms. That his invention was the greatest boon genius could bestow upon a Maritime Empire is a truth universally felt, and which no person is better qualified to appreciate than your Majesty. It is a proud reflection for Britain, that she does not owe to a stranger the creation of that intellectual aid which renders your Majesty's Fleets as free and fearless in Navigation as they have ever been in Battle.

To such considerations alone am I entitled to attribute your Majesty's condescension in accepting of this work.

I have the honour to remain,
Your Majesty's humble and devoted
Subject and Servant,
MARK NAPIER.

Mathematics and contemporary industry . . .

Some fifteen years have passed since Thornton C. Fry dramatically forecast the role of the industrial mathematician.[4] From time to time, scattered here and there in the literature, one gathers further evidence of the impact of mathematics on the complexities of contemporary industry. In 1952 the *Society for Industrial and Applied Mathematics* was founded in Philadelphia, having been anticipated in 1949 by the *Industrial Mathematics Society* of Detroit, Michigan. During the past decade a number of notable new organizations have been created—among

[4] Thornton C. Fry, *Industrial Mathematics* (Bell Telephone Laboratories, June 1941), 38 pages.

them, *Operations Research Society, Association for Computing Machinery,* the *Institute of Management Sciences, Association of Machine Accountants,* and *Systems and Procedures Association.*

We believe that teachers of mathematics should become more familiar with (1) the nature of the applications of mathematics to modern industry, and (2) the personnel needs of industry which involve mathematics, and, as a corollary, the mathematical training and competence required of such personnel. Indeed, this is one of the pressing problems of our time. We therefore call our readers' attention to a recent sixteen-page pamphlet, *Mathematics in an Industrial Economy,* published in 1955 by the Industrial Mathematics Society, Detroit, Michigan. The pamphlet highlights these and kindred matters, and we take the liberty of abstracting some observations from the first sections of the pamphlet in question.

Perhaps the outstanding feature of our culture is the dominantly industrial character of its economy. We leave it to the social scientists to describe its growth while we try to examine some of the recent changes as they affect science and technology. Specifically we will be concerned with the new and central role of mathematics in all systematic inquiry and conduct . . . .

Better understanding of the behavior of physical systems and new computational techniques have made it possible to formulate and solve problems more realistically. It is no longer necessary, as a rule, to idealize the problem to obtain only an approximate expression for it. This often means that the problem need not be linearized but can be considered with all of the nonlinearities present in the physical problem. Thus, studies in elasticity and plasticity, for example, take into account changes in internal energy due to temperature, often yielding formulations of considerable complexity . . . .

Among the great advances in electrical engineering has been the application of analytical and statistical techniques to complex networks and devices relating to communication, power generation and distribution, automatic control and computation . . . . The fields of automatic control and information theory are to a large extent new applications of quantitative methods. The successful formulations of the linear feedback principle as well as the significant advances in the study of nonlinear systems have made it possible to synthesize automatic control mechanisms of great complexity, range, and sensitivity. This has been achieved primarily by judicious analysis of the mathematical models describing these complex feedback systems.

Major research efforts of the economists have been directed toward the determination of quantitative interrelations among various sectors of industry, the construction of models of the dynamic processes in the national economy and the prediction of the behavior of large economic complexes . . . . In the industrial field the use of statistics has witnessed an unprecedented expansion, especially in the form of statistical quality control and in the design and analysis of experiments. Progress likewise has been made in applications of sampling theory and factor analysis to life expectancy of products.

The growing volume of knowledge and the complexity of our social organization have made necessary an incredible expansion of the application of logical and quantitative methods. Here the automatic computer has greatly facilitated and extended the usefulness of mathematics. The electronic computers and data processing systems make logical and analytical investigations possible on a scale undreamed of only a few decades ago. The Industrial Revolution, as it embraced the mechanization of operations, was only a forerunner to the much greater form of intelligence, control, and prediction that man will acquire in the ordering of his life by use of the high speed automatic computing and information processing devices.

What this means for the place of mathematics in education, and for the responsibility of those engaged in counseling young people, is perhaps well summed up in the following passage from the same pamphlet:

An eminent industrial mathematician once observed that industry's needs of mathematical talent were best served by superposing mathematical education upon a thorough training in physics and engineering.

In recent years the emphasis has changed to the degree that mathematicians as such are being welcomed into industrial organizations, with preferences frequently expressed for persons whose education and experience are so extensively mathematical as to preclude any opportunity for appreciable training in physics or engineering. With the entry of mathematicians into industry, the need for a better command of mathematics on the part of engineers and physicists has increased in order that the latter would be able to communicate with the former.

Since mathematics is being accepted as one of the industrial sciences, its symbolism and concepts appear more generally and frequently. Thus communications and procedures become more mathematical in nature, requiring better understanding of mathematics even on lower

levels of personnel.

With the need of greater mathematical competence of more people, the demands on better and more efficient education in this field are clear. Teachers, on all levels, must have a good command of this important subject and they must have adequate training in its teaching. They must also have a satisfactory grasp of its ubiquitous, ever expanding sphere of utility.

School administrators and industrial executives must likewise have a thorough appreciation of the place of mathematics in the overall education of an individual, and in the effective performance of his duties as an employee. Especially is it important that personnel managers take proper cognizance of mathematical preparation in recruitment, upgrading and placement. It is also of paramount importance that job classifications of mathematicians be established in each industry and business where they are warranted. The Federal Civil Service Commission has already set up such classifications, with appropriate job descriptions. The establishment of the classifications of mathematicians is imperative if we are to persuade young people to major in the subject.

What's New?

BOOKS

Secondary

A First Course in Algebra (revised edition), N. J. Lennes, J. W. Maucker, John J. Kinsella, New York, The Macmillan Company, 1957. Cloth, xviii+558 pp., $3.48.

A Second Course in Algebra (revised edition), N. J. Lennes, J. W. Maucker, John J. Kinsella, New York, The Macmillan Company, 1957. Cloth, xix+476 pp., $3.80.

Second-Year Algebra (revised edition), Herbert E. Hawkes, William A. Luby, Frank C. Touton, Boston, Ginn and Company, 1956. Cloth, viii+526 pp., $3.16.

Solid Geometry, Rolland R. Smith and James F. Ulrich, New York, World Book Company, 1957. Cloth, x+266 pp., $2.88.

College

Advanced Calculus, R. Creighton Buck, New York, McGraw-Hill Book Company, 1956. Cloth, viii+423 pp., $8.50.

Applied Analysis, Cornelius Lanczos, Englewood Cliffs, New Jersey, Prentice-Hall, 1956. Cloth, xx+539 pp., $9.00.

Brief Analytic Geometry (3rd edition), Thomas E. Mason, Clifton T. Hazard, Boston, Ginn and Company, 1957. Cloth, ix+229 pp., $3.50.

Electric Computers, edited by T. E. Ivall, New York, Philosophical Library, 1956. Cloth, viii+167 pp., $10.00.

BOOKLETS

A Collection of Cross-Number Puzzles. J. Weston Walch, Publisher, Box 1075, Portland, Maine. Booklet written by Louis Grant Brandes, suitable for grades 7–9. Teacher Edition (226 pp.), $2.50; Student Edition (156 pp.), $2.00.

A Plan for Living, Oscar G. Mayer, Chicago, Oscar Mayer Foundation, 1956. Cloth, 23 pp., free.

Educational Utilization of Masonite Peg-Board Panels and Fixtures, School Service Bureau, Masonite Corporation, 111 West Washington Street, Chicago 2, Illinois. 20-page illustrated booklet based upon a research project directed by the Stanford University School Planning Laboratory, free.

An aircraft parts maker has built an electronic gauge to test tiny parts delicate enough to measure the depth of a mouse footprint on a strand of wire.—*Taken from* PLANES, *Official Publication of the Aircraft Industries Association of America.*

A column of unofficial comment

Is research in mathematics education really necessary?

by John J. Kinsella, New York University, New York, New York

Some readers may wonder why the question stated in the title of this piece is "really necessary." One answer was given by a reviewer of research in mathematics education to 1953. His conclusion was that there is "no decisive proof that any particular method of teaching (inductive, deductive, individual, group) or any particular philosophy of teaching will guarantee better results than any other method or philosophy, so far as achievement is concerned." A second reason for bringing up the question is found in the infinitesimal amount of reported research done on the teaching of college mathematics. Either such research is considered pointless because college teaching is a highly individual art, or because precollege programs have produced students who have learned how to learn mathematics no matter how they are taught.

Of course, there is more to research in mathematics education than research in methods of teaching and investigations into how students learn. Decisions about *what* to teach, *when* to teach the what, and *to whom* to teach the what, can, of course, be made by one individual consulting with himself or a small coterie, but it might be better to increase the number of competent observers and, in addition, to present and analyze the evidence in public view. Probably questions about *why* to teach the "what" should be assigned to some philosophers comfortably ensconced, but certainly not to a Keyser, a Russell, or a Whitehead.

Although men like Poincaré, Hadamard, and Polya have ably portrayed the importance of intuition and imagination in mathematical activity, hardly any member of our fraternity feels satisfied unless the deductive proof label is finally stamped on his productions. Since "transfer" among mathematicians and teachers of mathematics is automatic, it would seem to follow that in making decisions about what to teach, how to teach, when to teach, and to whom to teach, similar standards of proof would be applied. Although research in mathematics education, in contrast with research in mathematics, cannot yield proof of the degree of rigor and certainty found in modern algebra and set theory, we should, if for no other reason than to be consistent with our ideal of consistency, do all we can to maximize the probability that the content we select and the teaching methods we use are the best available at any given time.

On the other hand, good research in mathematics education is extremely difficult. Teacher and student variables not only refuse to take fixed values, despite our commands, but interact in unpredict-

able ways. Despite the contributions of statistics to the research undertaking, like the analysis of variance and covariance, factor analysis, discriminant analysis, and various nonparametric methods, many of the difficulties remain. After all, the value of statistical techniques is dependent on valid and reliable measurement of the variables themselves. There is little difficulty here with our "achievement tests," for the kind of achievement they measure is principally operational skill. However, if we seek measurement of understanding or of the ability to apply what is presumed to be understood, we will find some progress only in the field of elementary school arithmetic. Furthermore, it is doubtful that pencil-and-paper tests alone are sufficient to provide reliable evidence of understanding or of the ability to apply what is learned in contexts different from those in which the initial learning occurred.

Criticism of research in education, including mathematics education, is not unknown. It has been said that too much of such research is an attempt to prove what is already known. Yet when surveys are made of what is actually happening in classrooms throughout a wide geographic area of the "land of the free," it is difficult to find that the obvious is being put into practice to any large extent. It has been said that "the experimenter always wins." These critics find it convenient to ignore, or simply do not know about, the studies of Johnson, Swenson, Anderson, Sobel, Van Engen, and Gibbs; the honesty of these investigators is apparent, for they were magnanimous enough to report what they found even when the findings did not agree with their previously expressed anticipations. It has been said that too many investigators swear allegiance to the "law of the single variable"; it is said that what happens to one variable is studied to the neglect of other variables that obviously influence the results. These critics have apparently not been aware of the many studies in mathematics education which use analysis of variance, in which many variables are studied concurrently. It is said that too many studies are conducted by one investigator with one class in a given year in a given place with a given teacher. As long as such investigators do not claim universality for their results, they have at least had the courage, like pioneers, to initiate something. Until some foundation or other type of national organization contributes to research organization by co-ordinating research and providing for the repetition of research studies, it is better for such individual research workers to do something besides sitting on their hands. A final example of such criticism of research is that the results of different investigators are conflicting. This should not be surprising when the number of variables operating in the separate individual situations are taken into account. Careful review of such experiments would probably reveal that conditions varied from one experiment to another, as well as the precautions taken by the investigators.

Among the current problems of mathematics education are those of pruning the high school and college mathematics curricula so that they can be enriched with concepts of more importance; of integrating the content of mathematics so that it is tied together with some unifying ideas instead of being a scattering of isolated activities; of making deductive proof less a monopoly of the high school geometries and more the property of other branches of high school mathematics. There is a time for individual initiative and private pioneering in attacking these problems. There is also a time for setting up hypotheses, anticipating consequences, gathering relevant data in a co-operative public way, analyzing them and moving toward decisions soundly based. That is the research way.

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Descriptive Geometry, Steve M. Slaby, New York, Barnes and Noble, 1956. Paper, xiii +353 pp., $2.25.

This book, one of the College Outline Series, is especially suited to the needs of the student who is studying descriptive geometry by himself. It could be used profitably as an additional source of material for the student who is studying the traditional course in the subject. A tabulated bibliography of standard college texts is keyed to the contents of this manual. Another bibliography offers a quick reference table to standard textbooks for the student who is using this book as a text. The style is simple and uncomplicated. It is geared to the needs of the average college student and is well-organized. Drawings are unusually clear, uncluttered, and admirable in every way. Each of the ten chapters is followed by many practice problems, and four sections in an appendix are devoted to examples of descriptive geometry found in everyday engineering. At the price, $2.25, it would make a welcome addition to the shelves of the high school teacher of solid geometry.—*Ernest R. Ranucci, Weequahic High School, Newark, New Jersey.*

Intermediate Algebra (with answers), William G. Shute, William E. Kline, William W. Shirk, LeRoy M. Willson, New York, American Book Company, 1956. Cloth ix+486 pp., $3.40.

Algebra and Its Use (Book II), Ethel L. Grove, Anne M. Mullikin, Ewart L. Grove, New York, American Book Company, 1956. Cloth, iv+538 pp., $3.40.

One would expect two books on the same subject, issued by the same publisher in the same year, to reflect a contrast in some significant respect. Although there are many minor differences, the books are basically very similar. Both cover the traditional subject matter of the second course in algebra adequately. They reflect much the same strengths and weaknesses. The format of both books is excellent. Color printing is employed in each, but not to very very good advantage in either. The use of color, blue in the case in *Algebra and Its Uses* and red in *Intermediate Algebra*, is used primarily with cartoons, a generous supply of which appears in each. The cartoons may appeal to high school pupils, but are in most cases of doubtful value as learning aids. The other uses of color seem to be without any particular plan or motive. In *Intermediate Algebra*, rules are boxed off for emphasis and ease of location. They are usually boxed in black but occasionally in red. Line drawings and graphs are handled similarly. *Algebra and Its Uses* employs less color, but it is used in the same way. Both books contain good chapter summaries and cumulative tests.

Distinctive features of *Intermediate Algebra* include "flashbacks" which give interesting and pertinent bits of historical background, "Doings of Dimbo" to point up typical pupil mistakes, and "Just For Fun" problems. All of these features can be used to advantage in the hands of a skillful teacher.

The authors of *Algebra and Its Use* have made a commendable effort to provide the "why" for the rules that are developed. They have also provided a wealth of well-stated problems.

Both books contain ample practice-exercise material. In *Intermediate Algebra* many of the stated problems are presented by types—which is regrettable in the reviewer's judgment but which probably reflects the "follow-the-rule" philosophy of the book. Beyond the usual coverage, *Intermediate Algebra* includes work with the oblique triangle, permutations, combinations, and probability, as well as an introduction to analytic geometry and statistics.

Neither text reflects much of the spirit of modern mathematics. One could not discover from either that algebra is a postulational system which is in many ways superior to geometry as a vehicle for deductive reasoning. Basic assumptions are not always identified as such. It is frequently difficult to distinguish between descriptions and definitions, the correctness of which are sometimes open to question. Vague and unprecise terminology is sometimes used.

On page 1 of *Intermediate Algebra* numbers are classified as *arithmetic numbers, literal numbers*, and *signed numbers*. It is not clear whether arithmetic numbers are the natural numbers or the positive reals. "A negative number is written always with a minus sign in front of the number." What about $2a$ if $a<0$? There is no attempt to define the integers in terms of the natural numbers. The discerning student is apt to be puzzled by such statements as "The expression $\pm 3a^4$ is read 'plus or minus $3a^4$,' and means $+3a^4$ and $-3a^4$." And "Likewise, if $(x+4)(x-3)=0$, then $x+4=0$ or $x-3=0$ or both $x+4$ and $x-3$ equal zero." How can $(x+4)$ and

$(x-3)$ both equal zero? If the first factor is zero, then $x=-4$ and the second factor must be -7. If the second factor is zero, then $x=3$ and the first factor must be $+7$.

Prime numbers are incorrectly defined on page 63. On page 47 we find the statement, "Factoring a quantity means finding two or more expressions whose product equals that quantity." Accordingly, one may say $\frac{17}{i}$, 6, and $\frac{-2}{17i}$ are factors of 12. If " . . . any expression whose value depends upon the value of x is a function of x," $f(x)=2$ must be meaningless symbols. *Transpose* and *cancel* are terms which are entirely unnecessary and may lead to confusion. If we must use them, then the student is entitled to a logically correct frame of reference in which to put them. Rather than calling cancelling an algebraic operation, why not point out that it is the process of replacing two inverse elements by the identity element relative to the operation involved? When we simplify $5+x-5$ by cancelling the fives, we are replacing the inverse elements 5, -5 with the identity element zero, or what amounts to the same thing, we are refusing to add 5 to x and then take 5 away. Similarly, when we replace $\frac{2ax}{3ay}$ with $\frac{2x}{3y}$ we replace $a\cdot\frac{1}{a}$ by the identity element 1, or we refuse to both multiply and divide $\frac{2x}{3y}$ by a. The concept of the identity element under multiplication could have been used to advantage in connection with the "Fundamental principle of fractions," p. 74. The limit concept is casually but abruptly injected into the picture with this statement, "As n increases without limit, r^n approaches zero as a limit." It is not pointed out to the pupil that the sum of an infinite convergent series is what it is by definition. That is to say, that the concepts of "sum" are not identical for finite and infinite series. The meanings of zero and negative exponents are "proved" by assuming the laws of exponents, which had been established for positive integral exponents (the only kind defined), must hold for all exponents. The classification of the complex number system, on pages 200 and 206, is confused. If "All numbers are either *real* or *imaginary*" and "An *imaginary* number is the indicated even root of a negative number," is the complex number $3+\sqrt{-5}$ real or imaginary? Since no attempt is made to state what a real number is, the above statements would require us to call $3+\sqrt{-5}$ a real number.

The apparent aversion on the part of the authors of *Algebra and Its Use* to the term *inverse* is puzzling. This is typical of much of the terminology in the book. For example, "To *divide out* means that one quantity divides into another quantity exactly one time, giving a quotient of 1." This seems an odd way to say that the ratio of two equal quantities is unity. Subtraction is referred to as "negative addition" in establishing the laws of signs for multiplication. Aside from the fact that most authors agree that 4×3 means $3+3+3+3$ rather than $4+4+4$, if we say that $(-4)(3)$ means three subtractions of -4, the minus sign in -3 is an operational sign and 3 is a natural number. The biggest objection to these derivations is the fact that there is no indication whether we are given definitions, proofs, or discussions. In discussing measurement no distinction is made between the concepts of accuracy and precision. The statement, "There is no such thing as a measurement which is exactly accurate," apparently means no measurement is free of error. If "To 'round off' a number means to replace some of its digits with zeros" then we have rounded off 7462 when we change it to 7062. If "Significant figures or digits are digits obtained by measurement," can we say $\sqrt{2}=1.414$ to four significant digits?

The authors assume that the quadratic function $f(x)$ and the quadratic equation $f(x)=0$ have the same graph. If the graph of an equation is the locus of points whose co-ordinates satisfy the equation, then the graph of $f(x)=0$ certainly is not the parabola obtained by graphing the function $f(x)$. The graph of the equation will be two vertical lines that intersect the parabola on the X axis.

It is readily granted that an objectionable statement looks worse out of context. Most of those cited would not do serious harm to the pupil when taken in their setting. The reviewer feels that both books are above average from the standpoint of high school algebra as it is usually taught. They are more than adequate from any standpoint as a source of exercises and problems. But the teacher whose goal it is to develop in the pupil a feel for what modern mathematics really is, to see and appreciate algebra as a beautiful logical structure, to see its relationship to the complex number system, to develop a modern conception of *variable*, *function*, etc., cannot rely much upon the texts. In fact they will at times be a detriment.—*J. Houston Banks, George Peabody College for Teachers, Nashville, Tennessee.*

Mathematics, Magic and Mystery, Martin Gardner, New York, Dover Publications, Inc., 1956. Paper, xii+176 pp., $1.00.

There are a surprising number of books from which you can learn either tricks or curious bits of mathematics: you could hardly get more entertainment from any of them at the price of this one, which combines both types.

The reader with a knowledge of algebra will find a good deal of satisfaction in translating the directions here into formulas, to see for himself how and why the tricks work. The nonmathematical reader will find the directions easily

understood and the outcomes remarkable.

The author evidently associates more with practitioners of magic than with professors of mathematics, but his mathematics is sound and varied, and not too profound for nonprofessional lovers of theory.

There are tricks based upon cards, dice, and coins, manipulations of strings and paper, geometrical mysteries (no fallacies), and interesting number theory.

The author gives satisfactory clues to explain why the tricks work, but does not go into too much detail to spoil the enjoyment of studying them out. Besides the mathematics of the tricks, advice is given about how to present them effectively. The point of view of the book is not that of the student in an ivory tower, but of one who enjoys displaying to an observer. The author's psychological skill is shown not only in advice about exhibition, but also as to how to guess what is probable in cases where the exhibitor is supposed to divine what is in the mind of the onlooker.

There are a large number of diagrams, serving their purposes very clearly, but the chief criticism of the book is that in putting it together the cuts are usually on the wrong pages, where they cannot be easily consulted while reading the text that relates to them. This is a very common fault in books that call for study: turning pages in the midst of following a reasoned passage is very frustrating. The excuse seems to be that it is cheaper to do it that way. Ancient books had their diagrams on pages at the end that could be folded out to go with any part of the text. But publishers should be brought to realize that any two open pages should contain what will make them intelligible.

Some of the card tricks involve a little manual skill, and some of the number tricks require a quick look or a ready memory; but on the whole, the mathematical basis of most of the tricks insures their automatic success in any hands.

The Möbius band type of trick is here in rich profusion, and other "topological tomfoolery." The properties of the Fibonacci numbers are exploited in a variety of ways. The widely known change from 64 to 65 is here accompanied by the far less widely known transformation from 64 to 63. Other, quite novel, ways of making bits of area seem to disappear occur in numerous variety. The "casting out" of one less than the radix is used in a variety of ways.

Unless you are quite widely read in this type of literature, you will find a great deal that is new to you, and will feel that you can not only learn some good tricks to entertain friends, but can get some good, substantial fun out of reading about them.

The book has a good index, frequent comments on the origins of different puzzles, and supplies references to a considerable number of books and magazine articles of like nature.—*William R. Ransom, Tufts University, Medford, Massachusetts.*

BOOKLETS

Calculus Quickly, William R. Ransom, Tufts University Press, Medford 55, Massachusetts. 60-page booklet, $1.50, postpaid.

In the 39 pages of text offered, Professor Ransom has touched upon the topics covered in about 300 pages of the usual textbook in calculus. In general, the differentiation and integration of polynomials, trigonometric circular and hyperbolic functions, and logarithmic functions are discussed very briefly and applied to problems.

The book was designed for use by high school seniors, by scientists, and by the general reader. It is well indexed and has a number of good tables. For a book of this size, it has an unusual number of exercises, these with answers. The language of the book is rather picturesque in spots, e.g., "terrace points" for extrema (page 15), "bowl" and "dome" to indicate the direction of concavity (page 15), "flats" for $f'=0$ and $f''=0$, with "flex" for $f'\neq 0$ and $f''=0$ (pages 15–16). "Radsin" is used to indicate $\sin x$, where x is expressed in radians. On page 16, there is a brief, but interesting, geometric proof of the theorem, $y''=1/R$ at a turning point, where R is the radius of the circle of curvature. The last chapter, Chapter 11, dealing with "Expansions," treats remainders in an elementary and interesting fashion.

However, the book cannot be recommended for any of the three categories mentioned. Professor Ransom has set himself an impossible task in attempting to condense so much mathematics into so few pages. To do this, he has had to leave out necessary definitions (such as "mean rate" on page 2) and every vestige of a developmental discussion, except in the last chapter. For example, limits are very inadequately treated; mathematical induction, used on page 8, is truncated beyond recognition; outside formulas are pulled out of a hat without an introduction, as on pages 16 and 29. The attempt to avoid mathematics in a book on mathematics led to as confusing a treatment of the problem of the stretched spring (Chapter 2) as anyone could fear to read. In addition, there are many questionable or even incorrect statements. On page 1: "The consideration of an interval is always the first step in a calculus problem." On page 2: "The shorter the interval, the less the speed varies in it." On pages 3 and 4, the author unfortunately gives the impression that a limit is found by substitution, and that the function must exist at the limit. On page 11, the author gives the impression that "fifty dollars at compound interest, compounded annually, at 6%, for x years" is an exponential function of the same kind as e^x, thus differentiable. On page 22: "The differential of a constant is zero, and so the integral of zero is a constant." Again, on page 22, $\int x^n dx$ forgets $n=-1$. On page 26, in a problem on maxima and minima, the endpoints are disregarded.—*Irving Allen Dodes, High School of Science, Brooklyn, New York.*

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

College board examinations and curriculum change

by Edwin C. Douglas, Taft School, Watertown, Connecticut

Editors' Note: Mr. Edwin C. Douglas has very graciously accepted your editors' invitation to discuss the role of College Entrance Examination Board tests in today's changing high school mathematics program. Mr. Douglas is eminently qualified to do this, since he is head of the mathematics department and assistant headmaster at Taft School, Watertown, Connecticut, and panel chairman of the College Board mathematics examining committees.

The mathematics achievement tests of the College Entrance Examination Board are thoroughly discussed in two pamphlets available to all teachers.[1] It would be presumptuous for the author to suppose he could add anything to the information in these excellent documents. However, it is quite possible that many readers of this column have not had the opportunity to become fully acquainted with College Board tests. At the risk of being repetitious, therefore, I shall first discuss the content of these examinations; following this I shall describe the relationship of these tests to curriculum change.

College board achievement tests

College Board sponsors three achievement tests in mathematics: Intermediate, Advanced, and Advanced Placement. Let us first consider the Intermediate and Advanced.

The Intermediate test covers plane geometry and elementary and intermediate algebra through quadratics; the Advanced test covers trigonometry, solid geometry, and advanced algebra, along with other topics customarily introduced in fourth year courses.

Both tests are designed to test ability to use what has been learned as well as to memorize and to manipulate. A few sample questions from both tests are listed below. They have been chosen from those published in the first pamphlet mentioned earlier, arranged according to the abilities they are designed to test, and numbered as in the pamphlet. A key to the correct answers will be found at the end of the last question.

I. *Manipulative skills and abilities*

Intermediate test

2. If $P=\dfrac{nk}{1-k}$, $k=(?)$

(A) $\dfrac{P}{P-n}$ (B) $\dfrac{P}{n+1}$ (C) $\dfrac{P}{n+P}$

(D) $\dfrac{P}{n-P}$ (E) $\dfrac{nP}{1-P}$

Advanced test

6. $\dfrac{1-i}{1+i}-\dfrac{1+i}{1-i}=(?)$ $(i=\sqrt{-1})$

(A) $-2i$ (B) -1 (C) 0 (D) 1 (E) $2i$

[1] See the first and second references in the bibliography at the end of the article.

II. *Knowledge and understanding of formulas, theorems, and mathematical terms*

Intermediate test

8. In Figure 1, $PQRST$ is an inscribed pentagon such that minor arc $PQ = 70°$. $\measuredangle R + \measuredangle T = (?)$

(A) 145° (B) 210° (C) 215°
(D) 290° (E) 430°

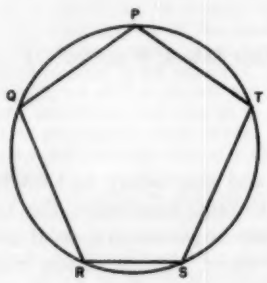


Figure 1

Advanced test

10. If x is an acute angle such that $\tan x = K/3$, $\sin x = (?)$

(A) $\frac{K}{3+K}$ (B) $\frac{3}{\sqrt{9-K^2}}$ (C) $\frac{K}{\sqrt{9-K^2}}$
(D) $\frac{3}{\sqrt{9+K^2}}$ (E) $\frac{K}{\sqrt{9+K^2}}$

III. *Ability to translate sentences into algebraic or graphic form, and conversely the ability to interpret algebraic or graphic representation*

Intermediate test

15. A taxicab company bases its fares on 55 cents for the first 3 miles and 10 cents for each additional mile or fraction of a mile. Which of the following is a formula for the fare F (in cents) for a trip of t miles, where t is an integer greater than 3?

(A) $F = 55 + 10(t-3)$
(B) $F = 55 + 10t$
(C) $F = 110 + 10t$
(D) $F = 55t + 10(t-3)$
(E) $F = 55t + 10$

Advanced test

19. A transport plane flies 150 miles on a course 60° west of south. It then takes a course 70° east of south until it is directly south of its starting point. Which of the following would give its distance (in miles) from the starting point?

(A) 150 sin 20°
(B) 150 sin 50° cos 70°
(C) 150 sin 50° sin 70°
(D) 150 sin 50°
(E) 150 sin 50° csc 70°

IV. *Ability to draw conclusions from data given*

Intermediate test

20. Given $\triangle PQR$ with median RS. Which of the following must be true?

I. RS is perpendicular to PQ
II. RS bisects $\angle QRP$
III. $\triangle PQR$ is a right triangle.
(A) None (B) Only I (C) Only II
(D) Only III (E) I, II, and III

Advanced test

22. If h, k, m, and n are positive numbers, k is greater than m, and n is greater than h, which of the following is (are) true?

I. $n+h$ may equal $k+m$
II. $k+h$ may equal $n+m$
III. $k+n$ may equal $m+h$
(A) Only I (B) Only I and II
(C) Only I and III (D) I, II, and III
(E) None

V. *Ability to recognize which facts or processes are necessary for the solution of a problem, and to use these accurately for solution*

Intermediate test

24. What is the greatest number of triangles with areas of 2 square inches each that can be cut from a rectangular sheet of paper 4 inches wide and 5 inches long?

(A) 5 (B) 8 (C) 10 (D) 20
(E) Cannot be determined

Advanced test

29. If $\log_{10} 5 = 0.70$, $\log_5 10 = (?)$

(A) 0.30 (B) 0.70 (C) 1.40
(D) 1.43 (E) 1.70

VI. *Ability to visualize forms and relationships in three-dimensional space and to apply knowledge of algebra, plane geometry, or trigonometry to them*

Advanced test

31. The distance between two parallel planes is d. The locus of points equidistant from these two planes and at distance $d/2$ from a line which lies in one of the planes is

(A) no points (B) one point
(C) one line (D) two lines (E) a circle

Answer key: 2. (C), 6. (A), 8. (C), 10. (E), 15. (A), 19. (E), 20. (A), 22. (B), 24. (C), 29. (D), 31. (C).

In addition, there is a separate publication (Reference No. 2) devoted to the examinations for Advanced Placement. The essay question section, now requiring two hours, probably most closely approaches in difficulty the Russian examinations recently discussed in this column. It is probably unfair to attempt a comparison of these examinations, since the Russian examinations seem to be devoted to algebra, geometry, and trigonometry while the Advanced Placement examinations of the College Entrance Examination Board are devoted to analytic geometry and the calculus. It is equally unfair to attempt a comparison between these examinations and the completely objective type intermediate and advanced achievement examinations which cover a good part of the material in the Russian examinations.

The philosophy behind the essay section of the Advanced Placement examination is somewhat different in that an attempt is made to find out what a candidate can do in two hours on reasonably difficult essay questions. This section of the student's examination answer paper is sent to the college mathematics department at which the candidate hopes to matriculate, thus enabling the department to see how the candidate attacks this type of question. It is an interesting liaison between school and college in which the actual written examination is the medium through which the liaison is established. Perhaps this test could be discussed in a later article.

It might be wise at this point to discuss rather briefly the makeup of the College Board Mathematics Examination Committees. There is one Committee of Examiners for the Intermediate and Advanced Achievement examinations, and a second one for the Advanced Placement examination. Supervising the work of both committees and acting as a general liaison man between the committees and the College Board is the Chairman of the Mathematics Panel.

The first committee is comprised of three college teachers, one high school teacher, and one independent secondary school teacher. Each is chosen by the College Board for his or her position in the field of the teaching of mathematics. Members are selected in an attempt to guarantee that the mathematics examinations will test the curriculum being taught, be suitable as a basis for college entrance, and be modern in nature. Decisions as to content and difficulty of these tests are the sole responsibility of this changing group of five mathematics teachers. The Committee for the Advanced Placement examinations is slightly larger and more heavily weighted in favor of the colleges, and justifiably so since it is essentially the work of the freshman year in college which is being examined.

College Board and curriculum change

A few years ago a group of Examiners in mathematics approached College Board authorities with the suggestion that there were rumblings going on in the ranks of the mathematics teachers at both the college and secondary level. They felt that there was dissatisfaction with both the existing

college freshman courses as well as with the secondary school mathematics curriculum. They were also aware of the fact that some freshman courses either had been changed or were going to be revised and that inevitably these changes would have to be reflected in the secondary school curriculum in mathematics.

However, the examiners were reluctant to experiment with or radically change the existing examinations lest the students be militated against in their college matriculation plans by being unprepared for a new type of examination covering material not universally taught. The examiners made the suggestion that the College Board might want to have on hand a series of adequately pre-tested examination questions designed to test what this small group felt might eventually be a contemporary secondary school curriculum in mathematics.

Thus the ancient debate was revived again. Does an examination merely test what is being universally taught? Does it therefore inadvertently set a curriculum, sometimes so firmly that it is difficult to change the long established syllabus? These were the major questions raised.

The College Board's position was eminently sound in that they felt that they were providing through The Educational Testing Service a series of tests which examined what was being taught at the moment in the secondary schools. The examiners rather felt that teachers tended to teach primarily what was certain to be examined and hence a status quo in the curriculum field was being set up by the rarely-changing tests.

The ultimate outcome of this discussion of the role of an examination in education was the establishment of the Commission on Mathematics by the College Board. The Commission is concerning itself with the college preparatory program in mathematics, and is working in close liaison with the Curriculum Committee of the National Council of Teachers of Mathematics and other similar groups. The goals and objectives of the Commission are clearly set forth in a pamphlet (Reference No. 3) which may be had on request. The significant thing as far as this column is concerned is that a study of a set of examinations led to the creation of a curriculum investigating body.

And now a word of caution to the teacher who may be worried about examination changes which may catch the student totally unprepared. Curriculum changes will have to be accepted reasonably universally, and will have to have been incorporated in textbooks before there will be any drastic change in the College Board tests. It is Board policy to announce these changes well in advance of any new test. Test items will have to be written and pre-tested using a reliable pretest population. This alone means that some group will have to be taught the suggested changes. The results of the pre-test will be used to select sound examination questions. When, and only when, all of these steps have been taken will there be a "new" examination. Fear of unfair tests, therefore, should not become a roadblock to progress in curriculum revision.

Bibliography

1. "A Description of the College Board Achievement Tests" (see sections devoted to mathematics), College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, New Jersey.
2. "Advanced Placement Program," College Entrance Examination Board, 425 West 17th Street, New York 27, New York.
3. "The Commission on Mathematics—Its Goals and Objectives," Commission on Mathematics, 425 West 117th Street, New York 27, New York.

Edited by Francis G. Lankford, Jr., Longwood College, Farmville, Virginia, and Joseph N. Payne, University of Wisconsin, Madison, Wisconsin

Teaching the concept of perimeter through the use of manipulative aids

by Jen Jenkins, Bethany College, Lindsborg, Kansas

Such terminology as "perimeter" should not be introduced until the pupils have had many meaningful experiences finding by measurement the distance around the outside of various figures. These distances might at first be called "outside measures." Tops of desks, frames of pictures, backs of books, surfaces of blackboards in the classroom, and other such surfaces should be measured to determine their outside measures.

To emphasize just what is being measured, the teacher might ask the pupils to do the following: "Place your finger at the lower left-hand corner of this rectangular surface. Now move your finger along the edge to the lower right-hand corner. Along what did your finger move? Write on the board the name of this. (Length, probably, will be the answer.) Measure the length to the nearest inch (or half-inch) and write the number beside the word 'length.' Now place your finger at the lower right-hand corner again and move it along the edge to the upper right-hand corner. Along what did you move your finger? (Width.) Write the word 'width' under the word 'length.' Measure the width and place the results beside the word 'width.' Since you started to move your finger around this rectangle, in how many directions has it moved? In what direction will it move now, if you move it along the next edge? Beside the 'length' and 'width' which you have written on the board, write the direction, such as 'to the right.' Now measure this third edge, writing the required information in its proper place on the blackboard. Then do the same for the fourth dimension."

The information on the blackboard might appear thus:

Length—6½ inches, to the right
Width—4 inches, up
Length—6½ inches, to the left
Width—4 inches, down

Addition should then be done, the answer being expressed somewhat in this manner:

Distance around the rectangle, 21 inches. The distance is two lengths and two widths.

A number of experiences of this sort should be made a part of the child's activities. Moving a toy car around a rectangular cardboard 5″ by 7″, representing a city park 5 blocks by 7 blocks, or walking around a rectangle that has been drawn on the floor, should lead pupils to make firm in their minds the idea that one moves in four different directions in measuring the outside distance of a rectangle and that one measures two lengths and two widths in so doing. Each time, the information pertinent to that rectangle should be placed on the board so that the child sees it again and again. Finally,

these different discoveries should assume a general form, such as this:

Finding Distance Around a Rectangle

1. We move in four different directions.
2. We change direction three different times.
3. We measure two lengths.
4. We measure two widths.
5. *Two lengths plus two widths is the distance around the rectangle.*

Using many different rectangles, the children should act out the last statement. When saying "two lengths," their fingers should move along one length and then along the other; when saying "two widths," the fingers should move along one width and then the other. Eventually, stating numbers for lengths and widths (such as two fives and two sevens are 24) without having any rectangle in sight will result in the pupil actually using the formula for the perimeter without knowing it is a formula. He will have had so many experiences with it that he can reason it out for himself.

When the time comes for changing the words to arithmetical shorthand, "two" becomes "2"; "length" becomes "L"; "plus" becomes "+"; "width" becomes "W"; "is" becomes "="; and "distance around a rectangle" becomes "P." Thus through sensory experience and discovery, the pupil makes his own rule. When he sees the rule in his textbook, he will regard it as an old friend, whom he learned to know throughout a period of time and through many pleasant experiences. A child who sees old friends in a textbook will cease to think of it as a strange land full of strangers; he will enjoy any contacts he has with this book that once might have held terrors for him. He will see $2L+2W=P$ and understand it, because he has painlessly created it.

An interesting and worth-while activity when studying distance around figures is that of forming different-sized rectangles from one given perimeter. The child uses cords having the same lengths and fastens them to wallboard in four places to make rectangles. Suppose 12 inches is the length of the cords being used. Rectangles two by four, one by five, and three by three could be formed, each of them having the perimeter 12 inches. If the cord is found to stretch too much—and it is likely to do that—use thin wire instead for this activity.

In studying the perimeter of a triangle, the pupil will note as he moves his finger along the sides that the movement is made in three different directions and that he changes directions two times. As he makes this movement, he should say, "The first side plus the second side plus the third side of the triangle is its outside measure (or perimeter)." After he has measured the sides he should add them together to obtain the numerical value of the perimeter. The generalization should be done in much the same manner as it was done with the perimeter of the rectangle, but it will go much faster.

When is Easter?

The great German mathematician, Gauss, translated the rules for the calculation of the date of Easter into simple mathematical terms. Perhaps some teachers of mathematics may be interested in the simplification of Gauss's rule as it applies in the twentieth century. Let N be the number formed by the last two digits of the number of the year. Divide N by 19 to get quotient x and discard the remainder. Divide N by 4 to get the quotient y and discard the remainder. Divide $19N+24-x$ by 30, to get remainder r, and divide $6N+3-y-r$ by 7 to get remainder s. Then Easter is on the $(22+r+s)$th day of March if $r+s \leq 9$, or the $(r+s-9)$th day of April if $r+s>9$, unless $r=28$ and $s=6$, when it is on the 18th of April. With $N=57$, we find April 21. Happy Easter!—*M. F. Smiley, University of Iowa* (*taken from the* Iowa Newsletter).

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D.C.

State representatives of The National Council of Teachers of Mathematics

Having state representatives has become almost a tradition with The National Council of Teachers of Mathematics. But this system of state representatives is more than a tradition. It is a vital and functional part of our organization.

Below is the current list of the state representatives. The representative in your state deserves your support and encouragement. He is an important link in our growing program.

The effectiveness of the state representatives can be measured best at the end of the school year, for at that time our most dependable membership count is made. At the present time the indications are that the rate of growth in memberships and subscriptions during the 1956–57 school year may prove to have been almost phenomenal. For this growth the state representatives, working with the Committee on Membership, will deserve much credit.

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A proposal to expand NEA services

One of the interesting features resulting from the location of the Office of the Council at the NEA headquarters is that those in the office have a chance to keep in touch with and sometimes contribute ideas to the over-all professional planning that is done. Furthermore, our status as a department of the NEA gives all members of the Council an interest in and a share in the responsibility for this planning, which may have a strong influence on the future of the teaching profession.

A frequent and serious topic of conversation at the NEA headquarters during recent months has been the proposed expanded program of services. History may have been made when the Representative Assembly of the NEA at the 1956 Convention in Portland, Oregon, approved the idea of a greatly expanded program of NEA services and recommended that the annual dues in the NEA be increased from $5.00 to $10.00 to support it. This proposal was discussed critically by more

than 3,000 delegates in 350 discussion circles. Although there were differences of opinion as to details, sentiment strongly favored the expanded program.

In the proposed program it has been suggested that services be increased in the following areas: Instruction, Legislation, Lay Relations, Professional Development and Welfare, Research, Communications, and Field Services. It has been generally agreed that such an expanded program could not be supported by the present dues of $5.00.

What are the implications of this proposed program for: 1. The future of the teaching profession in America? 2. The program and future of The National Council of Teachers of Mathematics?

The proposed program should challenge the interest and best thinking of all members of The National Council of Teachers of Mathematics. As the organized teaching profession, represented by the NEA, celebrates its one-hundredth birthday, it is especially appropriate that we give serious consideration to future plans and ideas.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of the MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH NEA AND NSTA
July 1, 1957
Philadelphia, Pennsylvania
M. H. Ahrendt, 1201 Sixteenth Street, N. W. Washington 6, D. C.

SUMMER MEETING
August 19–21, 1957
Carleton College, Northfield, Minnesota
Margaret Linster, St. Louis Park High School, Minneapolis 16, Minnesota, or Kenneth O. May, Carleton College, Northfield, Minn.

ANNUAL MEETING
April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

Other professional dates

Sixth Annual Meeting of the Pennsylvania Council of Teachers of Mathematics
April 27, 1957
Bucknell University, Lewisburg, Pennsylvania
Joy E. Mahachek, State Teachers College, Indiana, Pennsylvania

Eighth Annual Conference of the Michigan Council of Teachers of Mathematics
May 3–5, 1957
St. Mary's Lake, Battle Creek, Michigan
Margaret Lingo, Pierce Junior High School, Grosse Pointe, Michigan

Annual Meeting of the Association of Mathematics Teachers of New York State
May 10–11, 1957
Hotel Syracuse, Syracuse, New York
Randolph S. Gardner, New York State College for Teachers, Albany, New York

Women's Mathematics Club of Chicago and Vicinity
May 11, 1957
Tearoom of Mandel Brothers Department Store, Chicago, Illinois
Ruth Woerner, 11715 South 82nd Court, Palos Park, Illinois

Chicago Elementary Teachers Mathematics Club
May 13, 1957
Toffenetti's Restaurant, 65 West Monroe Street, Chicago, Illinois
Genevieve E. Johnson, Volta School, Chicago Illinois

Eighth Annual Mathematics Institute of Louisiana State University
June 23–29, 1957
Louisiana State University, Baton Rouge, Louisiana
Houston T. Karnes, Louisiana State University, Baton Rouge, Louisiana

Shell Merit Fellowship Program for Mathematics and Science Teachers
July 1–August 11, 1957
Cornell University, Ithaca, New York
Phillip G. Johnson, Cornell University, Ithaca, New York

Ninth Annual Institute of the Association of Teachers of Mathematics in New England
August 21–28, 1957
Dartmouth College, Hanover, New Hampshire
Alma A. Sargent, 125 North State Street, Concord, New Hampshire

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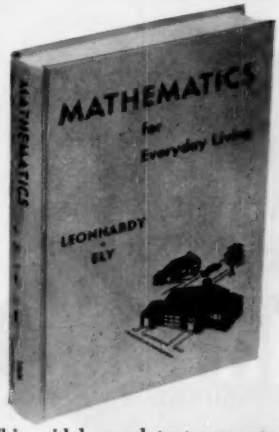

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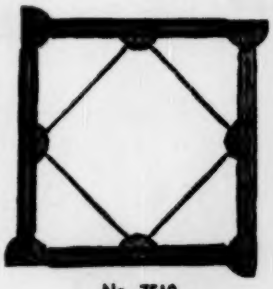

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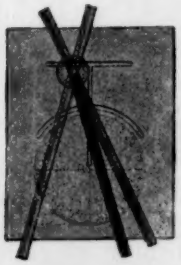

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